CHAPTER THIRTEEN

Solutions for Section 13.1

Exercises

1. $4\vec{i} + 2\vec{j} - 3\vec{i} + \vec{j} = \vec{i} + 3\vec{j}$

2. $\vec{i} + 2\vec{j} - 6\vec{i} - 3\vec{j} = -5\vec{i} - \vec{j}$

3. $-4\vec{i} + 8\vec{j} - 0.5\vec{i} + 0.5\vec{k} = -4.5\vec{i} + 8\vec{j} + 0.5\vec{k}$

4. $(0.9\vec{i} - 1.8\vec{j} - 0.02\vec{k}) - (0.6\vec{i} - 0.05\vec{k}) = 0.3\vec{i} - 1.8\vec{j} + 0.03\vec{k}$

5. $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}.$

6. $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$

7. $\|\vec{v}\| = \sqrt{1.2^2 + (-3.6)^2 + 4.1^2} = \sqrt{31.21} \approx 5.6.$

8. $\|\vec{v}\| = \sqrt{7.2^2 + (-1.5)^2 + 2.1^2} = \sqrt{58.5} \approx 7.6.$

9. $5\vec{b} = 5(-3\vec{i} + 5\vec{j} + 4\vec{k}) = -15\vec{i} + 25\vec{j} + 20\vec{k}.$

10. $\vec{a} + \vec{z} = (2\vec{j} + \vec{k}) + (\vec{i} - 3\vec{j} - \vec{k}) = (0+1)\vec{i} + (2-3)\vec{j} + (1-1)\vec{k} = \vec{i} - \vec{j}$

11. $2\vec{c} + \vec{x} = 2(\vec{i} + 6\vec{j}) + (-2\vec{i} + 9\vec{j}) = (2\vec{i} + 12\vec{j}) + (-2\vec{i} + 9\vec{j}) = (2-2)\vec{i} + (12+9)\vec{j} = 21\vec{j}.$

12. $\|\vec{z}\| = \sqrt{(1)^2 + (-3)^2 + (-1)^2} = \sqrt{1+9+1} = \sqrt{11}.$

13. $\|\vec{y}\| = \sqrt{(4)^2 + (-7)^2} = \sqrt{16+49} = \sqrt{65}.$

14.

$$\begin{aligned}
2\vec{a} + 7\vec{b} - 5\vec{z} &= 2(2\vec{j} + \vec{k}) + 7(-3\vec{i} + 5\vec{j} + 4\vec{k}) - 5(\vec{i} - 3\vec{j} - \vec{k}) \\
&= (4\vec{j} + 2\vec{k}) + (-21\vec{i} + 35\vec{j} + 28\vec{k}) - (5\vec{i} - 15\vec{j} - 5\vec{k}) \\
&= (-21-5)\vec{i} + (4+35+15)\vec{j} + (2+28+5)\vec{k} = -26\vec{i} + 54\vec{j} + 35\vec{k}.
\end{aligned}$$

15. (a) See Figure 13.1.
(b) $\|\vec{v}\| = \sqrt{5^2 + 7^2} = \sqrt{74} = 8.602.$
(c) We see in Figure 13.2 that $\tan\theta = \frac{7}{5}$ and so $\theta = 54.46°$.

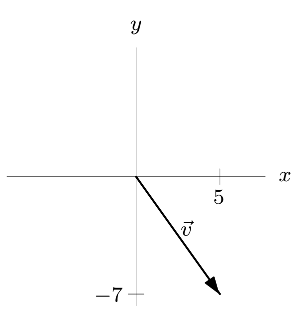

Figure 13.1

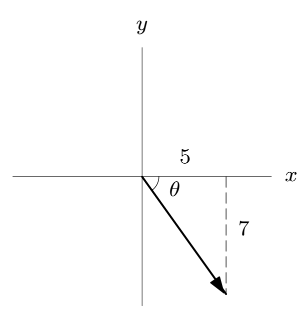

Figure 13.2

16. Resolving $\vec{v}$ into components gives $\vec{v} = 8\cos(40°)\vec{i} - 8\sin(40°)\vec{j} = 6.13\vec{i} - 5.14\vec{j}$. Notice that the component in the $\vec{j}$ direction must be negative.

17. $\vec{a} = -2\vec{j}, \vec{b} = 3\vec{i}, \vec{c} = \vec{i} + \vec{j}, \vec{d} = 2\vec{j}, \vec{e} = \vec{i} - 2\vec{j}, \vec{f} = -3\vec{i} - \vec{j}.$

18. The vector we want is the displacement from Q to P, which is given by

$$\vec{QP} = (1-4)\vec{i} + (2-6)\vec{j} = -3\vec{i} - 4\vec{j}$$

19. $\vec{a} = \vec{b} = \vec{c} = 3\vec{k}$, $\vec{d} = 2\vec{i} + 3\vec{k}$, $\vec{e} = \vec{j}$, $\vec{f} = -2\vec{i}$

20. $\vec{u} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = -\vec{i} + 2\vec{k}$.

Problems

21. To determine if two vectors are parallel, we need to see if one vector is a scalar multiple of the other one. Since $\vec{u} = -2\vec{w}$, and $\vec{v} = \frac{1}{4}\vec{q}$ and no other pairs have this property, only $\vec{u}$ and $\vec{w}$, and $\vec{v}$ and $\vec{q}$ are parallel.

22. The length of the vector $\vec{i} - \vec{j} + 2\vec{k}$ is $\sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$. We can scale the vector down to length 2 by multiplying it by $\frac{2}{\sqrt{6}}$. So the answer is $\frac{2}{\sqrt{6}}\vec{i} - \frac{2}{\sqrt{6}}\vec{j} + \frac{4}{\sqrt{6}}\vec{k}$.

23. (a) The displacement from P to Q is given by

$$\overrightarrow{PQ} = (4\vec{i} + 6\vec{j}) - (\vec{i} + 2\vec{j}) = 3\vec{i} + 4\vec{j}.$$

Since

$$\|\overrightarrow{PQ}\| = \sqrt{3^2 + 4^2} = 5,$$

a unit vector $\vec{u}$ in the direction of $\overrightarrow{PQ}$ is given by

$$\vec{u} = \frac{1}{5}\overrightarrow{PQ} = \frac{1}{5}(3\vec{i} + 4\vec{j}) = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

(b) A vector of length 10 pointing in the same direction is given by

$$10\vec{u} = 10(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}) = 6\vec{i} + 8\vec{j}.$$

24. Since the component of $\vec{v}$ in the $\vec{i}$-direction is 3, we have $\vec{v} = 3\vec{i} + b\vec{j}$ for some b. Since $\|\vec{v}\| = 5$, we have $\sqrt{3^2 + b^2} = 5$, so $b = 4$ or $b = -4$. There are two vectors satisfying the properties given: $\vec{v} = 3\vec{i} + 4\vec{j}$ and $\vec{v} = 3\vec{i} - 4\vec{j}$.

25. (a) The displacement vectors are:

$$\text{From the submarine to the ship} = -2\vec{i} + 3\vec{j} + 6\vec{k}$$
$$\text{From the helicopter to the ship} = 2\vec{i} + 2\vec{j} - 10\vec{k}$$
$$\text{From the submarine to the helicopter} = -4\vec{i} + \vec{j} + 16\vec{k}$$

(b) The distance between the submarine and the ship is $\sqrt{49} = 7$. The distance between the helicopter and the ship is $\sqrt{108} = 10.39$. The distance between the submarine and the helicopter is $\sqrt{273} = 16.52$. The submarine and the ship are the closest.

26. We get displacement by subtracting the coordinates of the origin $(0,0,0)$ from the coordinates of the cat $(1,4,0)$, giving Displacement $= (1-0)\vec{i} + (4-0)\vec{j} + (0-0)\vec{k} = \vec{i} + 4\vec{j}$.

27. We get displacement by subtracting the coordinates of the bottom of the tree, $(2,4,0)$, from the coordinates of the squirrel, $(2,4,1)$, giving:

$$\text{Displacement} = (2-2)\vec{i} + (4-4)\vec{j} + (1-0)\vec{k} = \vec{k}.$$

28.

$$\begin{aligned}\text{Displacement} &= \text{Cat's coordinates} - \text{Bottom of the tree's coordinates} \\ &= (1-2)\vec{i} + (4-4)\vec{j} + (0-0)\vec{k} = -\vec{i}.\end{aligned}$$

29.

$$\begin{aligned}\text{Displacement} &= \text{Squirrel's coordinates} - \text{Cat's coordinates} \\ &= (2-1)\vec{i} + (4-4)\vec{j} + (1-0)\vec{k} = \vec{i} + \vec{k}.\end{aligned}$$

30. (a) True, since vectors $\vec{c}$ and $\vec{f}$ point in the same direction and have the same length.
(b) False, since vectors $\vec{a}$ and $\vec{d}$ point in opposite directions. We have $\vec{a} = -\vec{d}$.
(c) False, since $-\vec{b}$ points in the opposite direction to $\vec{b}$, the vectors $-\vec{b}$ and $\vec{a}$ are perpendicular.
(d) True. The vector $\vec{f}$ can be "moved" to point directly up the z-axis.
(e) True. We move in the positive x-direction following vector $\vec{a}$ and then in the positive y-direction following vector $-\vec{b}$. The resulting sum is the vector $\vec{e}$.
(f) False, vector $\vec{d}$ is the negative of the vector $\vec{g} - \vec{c}$. It is true that $\vec{d} = \vec{c} - \vec{g}$.

31.

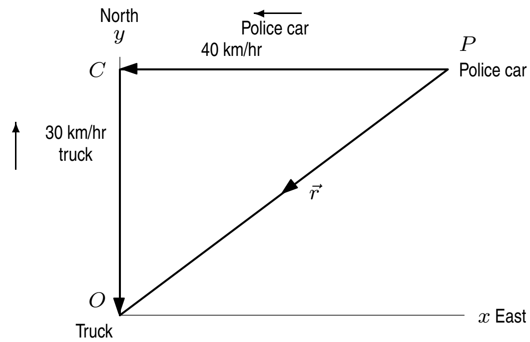

Figure 13.3

Since both vehicles reach the crossroad in exactly one hour, at the present the truck is at O in Figure 13.3; the police car is at P and the crossroads is at C. If $\vec{r}$ is the vector representing the line of sight of the truck with respect to the police car.

$$\vec{r} = -40\vec{i} - 30\vec{j}$$

32. In Figure 13.4 let O be the origin, points A, B, and C be the vertices of the triangle, point D be the midpoint of $\overline{BC}$, and Q be the point in the line segment $\overline{DA}$ that is $\frac{1}{3}|DA|$ away from D.

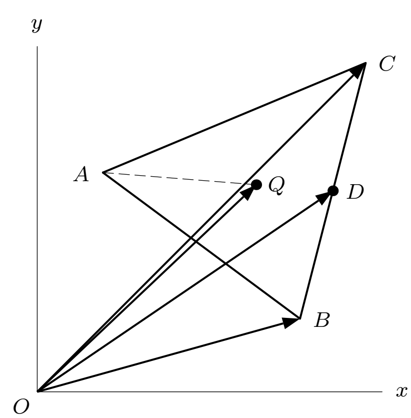

Figure 13.4

From Figure 13.4 we see that

$$\begin{aligned}\overrightarrow{OQ} = \overrightarrow{OD} + \overrightarrow{DQ} &= \overrightarrow{OD} + \frac{1}{3}\overrightarrow{DA} \\ &= \overrightarrow{OD} + \frac{1}{3}(\overrightarrow{OA} - \overrightarrow{OD}) \\ &= \overrightarrow{OD} + \frac{1}{3}\overrightarrow{OA} - \frac{1}{3}\overrightarrow{OD} \\ &= \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}.\end{aligned}$$

Because the diagonals of a parallelogram meet at their midpoint, and $2\overrightarrow{OD}$ is a diagonal of the parallelogram formed by $\overrightarrow{OB}$ and $\overrightarrow{OC}$, we have:

$$\overrightarrow{OD} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}),$$

so we can write:

$$\overrightarrow{OQ} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\left(\frac{1}{2}\right)(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

Thus a vector from the origin to a point $\frac{1}{3}$ of the way along median AD from D, the midpoint, is given by $\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$.

In a similar manner we can show that the vector from the origin to the point $\frac{1}{3}$ of the way along any median from the midpoint of the side it bisects is also $\frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. See Figure 13.5 and 13.6.

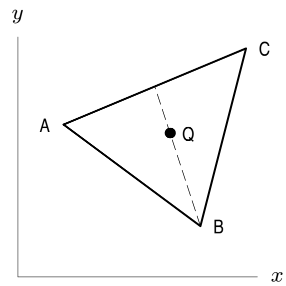

Figure 13.5

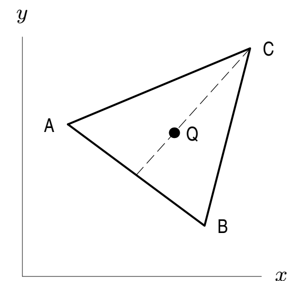

Figure 13.6

Thus the medians of a triangle intersect at a point $\frac{1}{3}$ of the way along each median from the side that each bisects.

33. We want to find an expression for a vector from the origin to a point that is $\frac{1}{4}$ of the way from a centroid to its opposite vertex.

In Figure 13.7 let O be the origin, A, B, C, and D be the vertices of a tetrahedron, P be the centroid of face BCD, and Q be the point on $\overline{PA}$ that is $|\frac{1}{4}\overline{PA}|$ away from P.

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ}\\ &= \overrightarrow{OP} + \frac{1}{4}\overrightarrow{PA}\\ &= \overrightarrow{OP} + \frac{1}{4}(\overrightarrow{OA} - \overrightarrow{OP})\\ &= \overrightarrow{OP} + \frac{1}{4}\overrightarrow{OA} - \frac{1}{4}\overrightarrow{OP}\\ &= \frac{1}{4}\overrightarrow{OA} + \frac{3}{4}\overrightarrow{OP}.\end{aligned}$$

In Problem 32 we showed that a vector from the origin to P, the centroid of a triangle, is

$$\overrightarrow{OP} = \frac{1}{3}(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}).$$

Substituting this into our expression for $\overline{OQ}$ gives

$$\begin{aligned}\overrightarrow{OQ} &= \frac{1}{4}\overrightarrow{OA} + \frac{3}{4}\left(\frac{1}{3}\right)(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})\\ &= \frac{1}{4}\overrightarrow{OA} + \frac{1}{4}(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})\\ &= \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}).\end{aligned}$$

In a similar manner we can show that a vector from the origin to a point $\frac{1}{4}$ of the way from the centroid of *any* face to its opposite vertex is $\frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})$. Thus, lines joining the centroid of each face to its opposite vertex all meet at a single point which is $\frac{1}{4}$ of the way from any centroid to its opposite face.

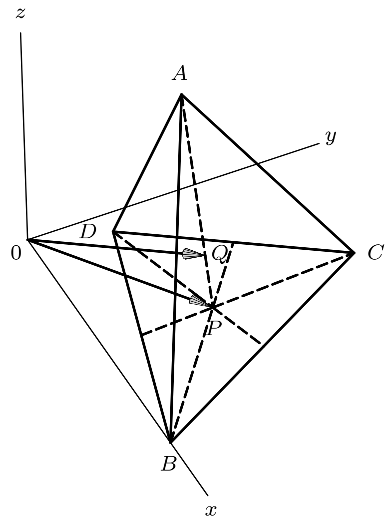

Figure 13.7

34. We must check that all the points are the same distance apart, i.e., the magnitude of the displacement vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{BA}$, $\overrightarrow{CB}$ and $\overrightarrow{CA}$ is the same. Here goes:

$$\begin{aligned}
\|\overrightarrow{OA}\| &= \|(2\vec{i}+0\vec{j}+0\vec{k}) - (0\vec{i}+0\vec{j}+0\vec{k})\| = \sqrt{2^2+0^2+0^2} = 2 \\
\|\overrightarrow{OB}\| &= \|(1\vec{i}+\sqrt{3}\vec{j}+0\vec{k}) - (0\vec{i}+0\vec{j}+0\vec{k})\| = \sqrt{1^2+(\sqrt{3})^2+0^2} = 2 \\
\|\overrightarrow{OC}\| &= \|(1\vec{i}+1/\sqrt{3}\vec{j}+2\sqrt{2/3}\vec{k}) - (0\vec{i}+0\vec{j}+0\vec{k})\| = \sqrt{1+1/3+4(2/3)} = 2 \\
\|\overrightarrow{BA}\| &= \|(2\vec{i}+0\vec{j}+0\vec{k}) - (1\vec{i}+\sqrt{3}\vec{j}+0\vec{k})\| = \sqrt{1+3+0} = 2 \\
\|\overrightarrow{CB}\| &= \|(1\vec{i}+\sqrt{3}\vec{j}+0\vec{k}) - (1\vec{i}+1/\sqrt{3}\vec{j}+2\sqrt{2/3}\vec{k})\| \\
&= \sqrt{0^2+(\sqrt{3}-1/\sqrt{3})^2+4(2/3)} = \sqrt{3-2+1/3+8/3} = 2 \\
\|\overrightarrow{CA}\| &= \|(2\vec{i}+0\vec{j}+0\vec{k}) - (1\vec{i}+1/\sqrt{3}\vec{j}+2\sqrt{2/3}\vec{k})\| = \sqrt{1+1/3+4(2/3)} = 2.
\end{aligned}$$

Solutions for Section 13.2

Exercises

1. Scalar

2. Scalar

3. The magnetic field is a vector because it has both a magnitude (the strength of the field) and a direction (the direction of the compass).

4. Temperature is measured by a single number, and so is a scalar.

5. Writing $\vec{P} = (P_1, P_2, \cdots, P_{50})$ where P_i is the population of the i-th state, shows that $\vec{P}$ can be thought of as a vector with 50 components.

6. We need to calculate the length of each vector.

$$\begin{aligned}
\|21\vec{i}+35\vec{j}\| &= \sqrt{21^2+35^2} = \sqrt{1666} \approx 40.8, \\
\|40\vec{i}\| &= \sqrt{40^2} = 40.
\end{aligned}$$

So the first car is faster.

7. In components, we have $\vec{v} = -40\cos(20°)\vec{i} - 40\sin(20°)\vec{j} = -37.59\vec{i} - 13.68\vec{j}$. Notice that both coefficients are negative. The components are $-37.59\vec{i}$ and $-13.68\vec{j}$.

8. In components, we have $\vec{v} = 10\cos(45°)\vec{i} - 10\sin(45°)\vec{j} = (5\sqrt{2})\vec{i} - (5\sqrt{2})\vec{j} = 7.07\vec{i} - 7.07\vec{j}$. Notice that the coefficient in the $\vec{j}$-direction must be negative. The components are $5\sqrt{2}\,\vec{i}$ and $-5\sqrt{2}\,\vec{j}$.

9. (a) If the car is going east, it is going solely in the positive x direction, so its velocity vector is $50\vec{i}$.
(b) If the car is going south, it is going solely in the negative y direction, so its velocity vector is $-50\vec{j}$.
(c) If the car is going southeast, the angle between the x-axis and the velocity vector is $-45°$. Therefore

$$\begin{aligned}\text{velocity vector} &= 50\cos(-45°)\vec{i} + 50\sin(-45°)\vec{j}\\ &= 25\sqrt{2}\vec{i} - 25\sqrt{2}\vec{j}.\end{aligned}$$

(d) If the car is going northwest, the velocity vector is at a 45° angle to the y-axis, which is 135° from the x-axis. Therefore:

$$\text{velocity vector } = 50(\cos 135°)\vec{i} + 50(\sin 135°)\vec{j} = -25\sqrt{2}\vec{i} + 25\sqrt{2}\vec{j}.$$

10. See Figure 13.8. Since

$$\tan\theta = \frac{18}{15},$$

we have

$$\theta = \arctan\left(\frac{18}{15}\right) = 50.194°.$$

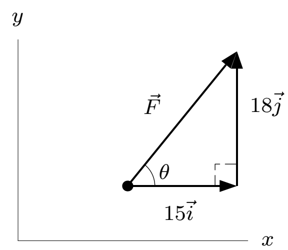

Figure 13.8

Problems

11. Let the velocity vector of the airplane be $\vec{V} = x\vec{i} + y\vec{j} + z\vec{k}$ in km/hr. We know that $x = -y$ because the plane is traveling northwest. Also, $\|\vec{V}\| = \sqrt{x^2+y^2+z^2} = 200$ km/hr and $z = 300$ m/min $= 18$ km/hr. We have $\sqrt{x^2+y^2+z^2} = \sqrt{x^2+x^2+18^2} = 200$, so $x = -140.8, y = 140.8, z = 18$. (The value of x is negative and y is positive because the plane is heading northwest.) Thus,

$$\vec{v} = -140.8\vec{i} + 140.8\vec{j} + 18\vec{k}.$$

12. (a) The velocity vector for the boat is $\vec{b} = 25\vec{i}$ and the velocity vector for the current is

$$\vec{c} = -10\cos(45°)\vec{i} - 10\sin(45°)\vec{j} = -7.07\vec{i} - 7.07\vec{j}.$$

The actual velocity of the boat is

$$\vec{b} + \vec{c} = 17.93\vec{i} - 7.07\vec{j}.$$

(b) $\|\vec{b} + \vec{c}\| = 19.27$ km/hr.
(c) We see in Figure 13.9 that $\tan\theta = \dfrac{7.07}{17.93}$, so $\theta = 21.52°$ south of east.

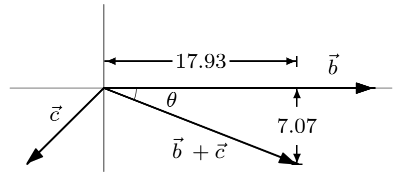

Figure 13.9

13. The velocity vector for the wind is $\vec{w} = 60\cos(45°)\vec{i} - 60\sin(45°)\vec{j} = 42.43\vec{i} - 42.43\vec{j}$. If the airplane is to head due east, then the component in the $\vec{j}$-direction of $\vec{p} + \vec{w}$ must be zero (where $\vec{p}$ represents the velocity vector for the airspeed of the plane.) Thus, we have $\vec{p} = A\vec{i} + 42.43\vec{j}$, for some value of A. Since the airplane is flying at an airspeed of 500 km/hr, we have

$$\begin{aligned} \|\vec{p}\| &= 500 \\ \sqrt{A^2 + 42.43^2} &= 500 \\ A &= 498.20. \end{aligned}$$

We have

$$\vec{p} = 498.20\vec{i} + 42.43\vec{j}.$$

This is the direction the plane should head in order to go due east. We use $\tan\theta = \frac{42.43}{498.20}$, so $\theta = 4.87°$. The plane should head $4.87°$ north of east. Since

$$\vec{p} + \vec{w} = 540.63\vec{i},$$

the airplane's speed relative to the ground is $\|\vec{p} + \vec{w}\| = 540.63$ km/hr.

14. The velocity vector of the plane with respect to the air has the form

$$\vec{v} = a\vec{i} + 80\vec{k} \text{ where } \|\vec{v}\| = 480.$$

(See Figure 13.10.) Therefore $\sqrt{a^2 + 80^2} = 480$ so $a = \sqrt{480^2 - 80^2} \approx 473.3$ km/hr. We conclude that $\vec{v} \approx 473.3\vec{i} + 80\vec{k}$.

The wind vector is

$$\begin{aligned} \vec{w} &= 100(\cos 45°)\vec{i} + 100(\sin 45°)\vec{j} \\ &\approx 70.7\vec{i} + 70.7\vec{j} \end{aligned}$$

The velocity vector of the plane with respect to the ground is then

$$\begin{aligned} \vec{v} + \vec{w} &= (473.3\vec{i} + 80\vec{k}) + (70.7\vec{i} + 70.7\vec{j}) \\ &= 544\vec{i} + 70.7\vec{j} + 80\vec{k} \end{aligned}$$

From Figure 13.11, we see that the velocity relative to the ground is

$$544\vec{i} + 70.7\vec{j}.$$

The ground speed is therefore $\sqrt{544^2 + 70.7^2} \approx 548.6$ km/hr.

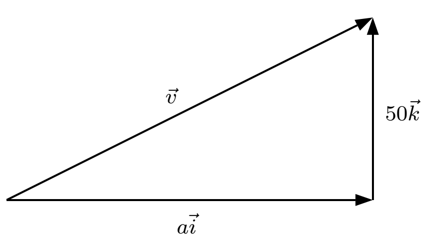

Figure 13.10: Side view

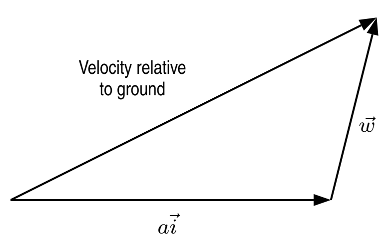

Figure 13.11: Top view

15. Let the x-axis point east and the y-axis point north. Since the wind is blowing from the northeast at a speed of 50 km/hr, the velocity of the wind is

$$\vec{w} = -50\cos 45°\vec{i} - 50\sin 45°\vec{j} \approx -35.4\vec{i} - 35.4\vec{j}.$$

Let $\vec{a}$ be the velocity of the airplane, relative to the air, and let ϕ be the angle from the x-axis to $\vec{a}$; since $\|\vec{a}\| = 600\text{ km/hr}$, we have $\vec{a} = 600\cos\phi\vec{i} + 600\sin\phi\vec{j}$. (See Figure 13.12.)

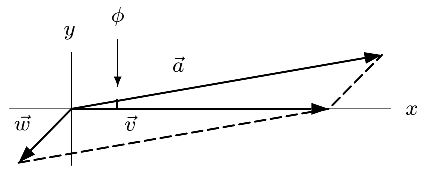

Figure 13.12

Now the resultant velocity, $\vec v$, is given by

$$\begin{aligned}\vec v &= \vec a + \vec w = (600\cos\phi\vec i + 600\sin\phi\vec j) + (-35.4\vec i - 35.4\vec j)\\ &= (600\cos\phi - 35.4)\vec i + (600\sin\phi - 35.4)\vec j.\end{aligned}$$

Since the airplane is to fly due east, i.e., in the x direction, then the y-component of the velocity must be 0, so we must have

$$\begin{aligned}600\sin\phi - 35.4 &= 0\\ \sin\phi &= \frac{35.4}{600}.\end{aligned}$$

Thus $\phi = \arcsin(35.4/600) \approx 3.4^\circ$.

16. (a) See Figure 13.13. Notice that the velocity vectors are tangent to the curve, they point in the direction of motion, and they are longer when the rocket is moving faster.

(b) If the rocket has a parachute, it comes down more slowly. The velocity vectors on the downward part of the graph are shorter for this rocket.

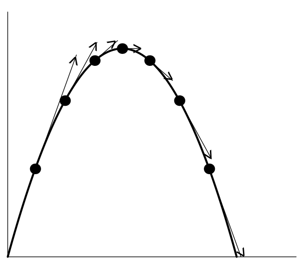

Figure 13.13

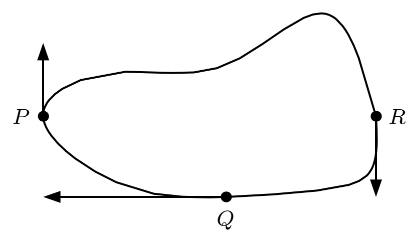

Figure 13.14

17. See Figure 13.14.

18. At the point P, the velocity of the car is changing the quickest; not in magnitude, but in direction only. The acceleration vector is therefore the longest at this point. The direction of the vector is directed in toward the center of the track because the difference in velocity vectors at nearby points is a vector pointing toward the center.

19. The total scores are out of 300 and are given by the total score vector $\vec v + 2\vec w$:

$$\begin{aligned}\vec v + 2\vec w &= (73, 80, 91, 65, 84) + 2(82, 79, 88, 70, 92)\\ &= (73, 80, 91, 65, 84) + (164, 158, 176, 140, 184)\\ &= (237, 238, 267, 205, 268).\end{aligned}$$

To get the scores as a percentage, we divide by 3, giving

$$\frac{1}{3}(237, 238, 267, 205, 268) \approx (79.00, 79.33, 89.00, 68.33, 89.33).$$

20. Since there are 16 ounces in a pound, we multiply the vector by $1/16$ to get $0.01875\vec i + 0.0125\vec j + 0.03125\vec k$ in dollars per ounce.

21. We want the total force on the object to be zero. We must choose the third force $\vec F_3$ so that $\vec F_1 + \vec F_2 + \vec F_3 = 0$. Since $\vec F_1 + \vec F_2 = 11\vec i - 4\vec j$, we need $\vec F_3 = -11\vec i + 4\vec j$.

22. Let the x-axis point east and the y-axis point north. We resolve the forces into components. Since the first force points 50° south of east with a force of 25 newtons, we have

$$\vec{F_1} = 25\cos(50^\circ)\vec{i} - 25\sin 50^\circ\vec{j} = 16.070\vec{i} - 19.151\vec{j}.$$

Since $\vec{F_1}$ lies in the fourth quadrant, the coefficient of $\vec{i}$ is positive and the coefficient of $\vec{j}$ is negative.

The second force points 70° north of west with a force of 60 newtons, so we have

$$\vec{F_2} = -60\cos(70^\circ)\vec{i} + 60\sin 70^\circ\vec{j} = -20.521\vec{i} + 56.382\vec{j}.$$

Since $\vec{F_2}$ lies in the second quadrant, the coefficient of $\vec{i}$ is negative and the coefficient of $\vec{j}$ is positive.

The third force must make the total force equal to zero, so we have

$$\begin{aligned}\vec{F_1} + \vec{F_2} + \vec{F_3} &= \vec{0}\\ \vec{F_3} &= -(\vec{F_1} + \vec{F_2})\\ &= -((16.070\vec{i} - 19.151\vec{j}) + (-20.521\vec{i} + 56.382\vec{j}))\\ &= -(-4.451\vec{i} + 37.231\vec{j})\\ &= 4.451\vec{i} - 37.231\vec{j}.\end{aligned}$$

The magnitude of this force is $\|\vec{F_3}\| = \sqrt{4.451^2 + 37.231^2} = 37.50$ newtons. The direction is $\arctan(37.231/4.451) = 83.20^\circ$ south of east.

23. The force exerted on the object from the first rope $\vec{F}_1 = 100\cos(30^\circ)\vec{i} + 100\sin(30^\circ)\vec{j} = 86.60\vec{i} + 50\vec{j}$ and the force exerted from the second rope is $\vec{F}_2 = 70\cos(80^\circ)\vec{i} - 70\sin(80^\circ)\vec{j} = 12.16\vec{i} - 68.94\vec{j}$. The sum of these two forces is $\vec{F}_1 + \vec{F}_2 = 98.76\vec{i} - 18.94\vec{j}$. See Figure 13.15. In order for the object to move vertically, the total force on the object must be in the form $\vec{F} = 0\vec{i} + 0\vec{j} + b\vec{k}$ for some b. Thus the force vector for the crane is

$$\vec{F}_c = -98.76\vec{i} + 18.94\vec{j} + b\vec{k}$$

for some b. To find b, we use the fact that $\|\vec{F}_c\| = 3000$. Thus,

$$\begin{aligned}\|\vec{F}_c\| &= 3000\\ \sqrt{(98.76)^2 + (18.94)^2 + b^2} &= 3000\\ b &= \pm 2998.31\end{aligned}$$

We use the positive value of b since we want the object to go up rather than down. The force exerted by the crane is

$$\vec{F}_c = -98.76\vec{i} + 18.94\vec{j} + 2998.31\vec{k}.$$

The total force acting on the object is $2998.31\vec{k}$, or 2998.31 newtons straight up.

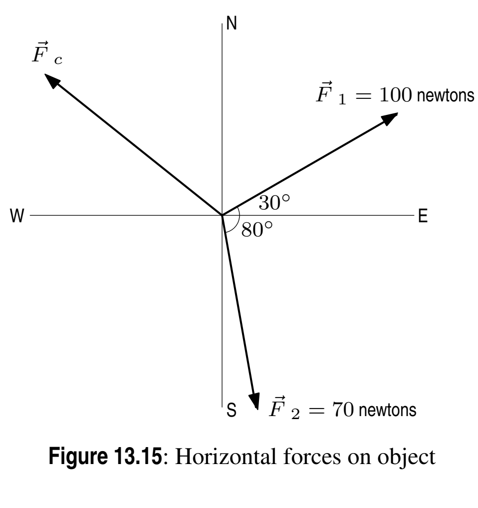

Figure 13.15: Horizontal forces on object

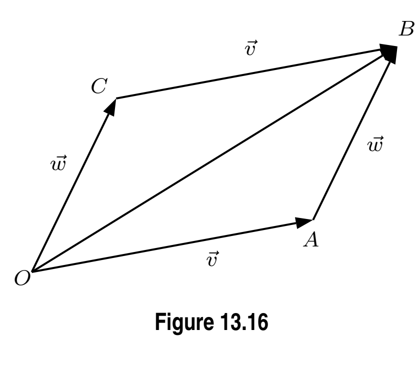

Figure 13.16

24. The vector $\vec{v} + \vec{w}$ is equivalent to putting the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ end-to-end as shown in Figure 13.16; the vector $\vec{w} + \vec{v}$ is equivalent to putting the vectors $\overrightarrow{OC}$ and $\overrightarrow{CB}$ end-to-end. Since they form a parallelogram, $\vec{v} + \vec{w}$ and $\vec{w} + \vec{v}$ are both equal to the vector $\overrightarrow{OB}$, we have $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

25.

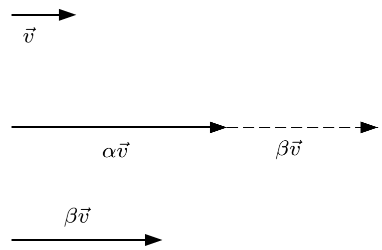

Figure 13.17

The vectors $\vec{v}$, $\alpha\vec{v}$ and $\beta\vec{v}$ are all parallel. Figure 13.17 shows them with $\alpha, \beta > 0$, so all the vectors are in the same direction. Notice that $\alpha\vec{v}$ is a vector α times as long as $\vec{v}$ and $\beta\vec{v}$ is β times as long as $\vec{v}$. Therefore $\alpha\vec{v} + \beta\vec{v}$ is a vector $(\alpha+\beta)$ times as long as $\vec{v}$, and in the same direction. Thus,

$$\alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v}.$$

26.

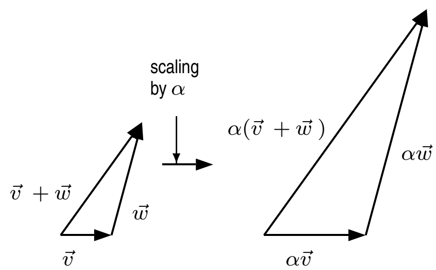

Figure 13.18

The effect of scaling the left-hand picture in Figure 13.18 is to stretch each vector by a factor of α (shown with $\alpha > 1$). Since, after scaling up, the three vectors $\alpha\vec{v}$, $\alpha\vec{w}$, and $\alpha(\vec{v}+\vec{w})$ form a similar triangle, we know that $\alpha(\vec{v}+\vec{w})$ is the sum of the other two: that is

$$\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}.$$

27. Assume $\alpha,\ \beta > 0$. The vector $\beta\vec{v}$ is in the same direction and β times as long as $\vec{v}$. The vector $\alpha(\beta\vec{v})$ is in the same direction and α times as long as $\beta\vec{v}$, and so is $\alpha\beta$ times as long as $\vec{v}$ and in the same direction as $\vec{v}$. Thus,

$$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}.$$

28. Since the zero vector has zero length, adding it to $\vec{v}$ has no effect.

29. According to the definition of scalar multiplication, $1 \cdot \vec{v}$ has the same direction and magnitude as $\vec{v}$, so it is the same as $\vec{v}$.

30. By Figure 13.19, the vectors $\vec{v} + (-1)\vec{w}$ and $\vec{v} - \vec{w}$ are equal.

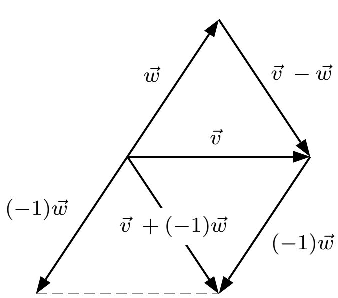

Figure 13.19

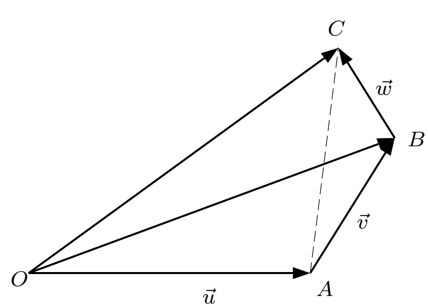

Figure 13.20

31. The vector $\vec{u} + \vec{v}$ is represented by $\overrightarrow{OB}$. The vector $(\vec{u} + \vec{v}) + \vec{w}$ is represented by $\overrightarrow{OB}$ followed by $\overrightarrow{BC}$, which is therefore $\overrightarrow{OC}$. Now $\vec{v} + \vec{w}$ is represented by $\overrightarrow{AC}$. So $\vec{u} + (\vec{v} + \vec{w})$ is $\overrightarrow{OA}$ followed by $\overrightarrow{AC}$, which is $\overrightarrow{OC}$. Since we get the vector $\overrightarrow{OC}$ by both methods, we know

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

Solutions for Section 13.3

Exercises

1. $\vec{a} \cdot \vec{y} = (2\vec{j} + \vec{k}) \cdot (4\vec{i} - 7\vec{j}) = -14.$

2. $\vec{c} \cdot \vec{y} = (\vec{i} + 6\vec{j}) \cdot (4\vec{i} - 7\vec{j}) = (1)(4) + (6)(-7) = 4 - 42 = -38.$

3. $\vec{a} \cdot \vec{b} = (2\vec{j} + \vec{k}) \cdot (-3\vec{i} + 5\vec{j} + 4\vec{k}) = (0)(-3) + (2)(5) + (1)(4) = 0 + 10 + 4 = 14.$

4. $\vec{a} \cdot \vec{z} = (2\vec{j} + \vec{k}) \cdot (\vec{i} - 3\vec{j} - \vec{k}) = (0)(1) + (2)(-3) + (1)(-1) = 0 - 6 - 1 = -7.$

5. $\vec{c} + \vec{y} = (\vec{i} + 6\vec{j}) + (4\vec{i} - 7\vec{j}) = 5\vec{i} - \vec{j}$, so

$$\vec{a} \cdot (\vec{c} + \vec{y}) = (2\vec{j} + \vec{k}) \cdot (5\vec{i} - \vec{j}) = -2.$$

6. $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y} = (\vec{i} + 6\vec{j}) \cdot (2\vec{j} + \vec{k}) + (2\vec{j} + \vec{k}) \cdot (4\vec{i} - 7\vec{j}) = 12 - 14 = -2.$

7. Since $\vec{a} \cdot \vec{b}$ is a scalar and $\vec{a}$ is a vector, the answer to this equation is a vector parallel to $\vec{a}$. We have

$$\vec{a} \cdot \vec{b} = (2\vec{j} + \vec{k}) \cdot (-3\vec{i} + 5\vec{j} + 4\vec{k}) = 0(-3) + 2(5) + 1(4) = 14.$$

Thus,

$$(\vec{a} \cdot \vec{b}) \cdot \vec{a} = 14\vec{a} = 14(2\vec{j} + \vec{k}) = 28\vec{j} + 14\vec{k}$$

8. Since $\vec{a} \cdot \vec{y}$ and $\vec{c} \cdot \vec{z}$ are both scalars, the answer to this equation is the product of two numbers and therefore a number. We have

$$\vec{a} \cdot \vec{y} = (2\vec{j} + \vec{k}) \cdot (4\vec{i} - 7\vec{j}) = 0(4) + 2(-7) + 1(0) = -14$$
$$\vec{c} \cdot \vec{z} = (\vec{i} + 6\vec{j}) \cdot (\vec{i} - 3\vec{j} - \vec{k}) = 1(1) + 6(-3) + 0(-1) = -17$$

Thus,

$$(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z}) = 238$$

9. Since $\vec{c} \cdot \vec{c}$ is a scalar and $(\vec{c} \cdot \vec{c})\vec{a}$ is a vector, the answer to this equation is another scalar. We could calculate $\vec{c} \cdot \vec{c}$, then $(\vec{c} \cdot \vec{c})\vec{a}$, and then take the dot product $((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a}$. Alternatively, we can use the fact that

$$((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a} = (\vec{c} \cdot \vec{c})(\vec{a} \cdot \vec{a}).$$

Since

$$\vec{c} \cdot \vec{c} = (\vec{i} + 6\vec{j}) \cdot (\vec{i} + 6\vec{j}) = 1^2 + 6^2 = 37$$
$$\vec{a} \cdot \vec{a} = (2\vec{j} + \vec{k}) \cdot (2\vec{j} + \vec{k}) = 2^2 + 1^2 = 5,$$

we have,

$$(\vec{c} \cdot \vec{c})(\vec{a} \cdot \vec{a}) = 37(5) = 185$$

10. A normal vector can be obtained from the coefficients: $\vec{n} = 2\vec{i} + \vec{j} - \vec{k}$.

11. Writing the equation in the form

$$3x + 4y - z = 7$$

shows that a normal vector is

$$\vec{n} = 3\vec{i} + 4\vec{j} - \vec{k}$$

12. The equation can be rewritten as

$$z - 5x + 10 = 15 - 3y$$
$$-5x + 3y + z = 5$$

so $\vec{n} = -5\vec{i} + 3\vec{j} + \vec{k}$.

13. Rewriting the equation as

$$2x - 2z = 3x + 3y$$

or

$$x + 3y + 2z = 0$$

tells us that a normal vector is

$$\vec{n} = \vec{i} + 3\vec{j} + 2\vec{k}.$$

Problems

14. (a) Increasing $||\vec{v}||$ increases $\vec{v} \cdot \vec{w}$ because $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$, and $\cos\theta$ is positive.
(b) Increasing θ decreases $\vec{v} \cdot \vec{w}$ because $\cos\theta$ is a decreasing function.

15. (a) Any multiple of $\vec{v}$ will work, for example, $8\vec{i} + 6\vec{j}$.
(b) Any vector $\vec{w}$ such that $\vec{v} \cdot \vec{w} = 0$ will work, such as $-3\vec{i} + 4\vec{j}$.

16.

$$\cos\theta = \frac{(\vec{i} + \vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} - \vec{k})}{\|\vec{i} + \vec{j} + \vec{k}\|\|\vec{i} - \vec{j} - \vec{k}\|} = \frac{(1)(1) + (1)(-1) + (1)(-1)}{\sqrt{1^1 + 1^2 + 1^2}\sqrt{1^2 + (-1)^2 + (-1)^2}}$$
$$= -\frac{1}{3}.$$

So, $\theta = \arccos(-\frac{1}{3}) \approx 1.91$ radians, or $\approx 109.5^\circ$.

17. Since $3\vec{i} + \sqrt{3}\vec{j} = \sqrt{3}(\sqrt{3}\vec{i} + \vec{j})$, we know that $3\vec{i} + \sqrt{3}\vec{j}$ and $\sqrt{3}\vec{i} + \vec{j}$ are scalar multiples of one another, and therefore parallel.
Since $(\sqrt{3}\vec{i} + \vec{j}) \cdot (\vec{i} - \sqrt{3}\vec{j}) = \sqrt{3} - \sqrt{3} = 0$, we know that $\sqrt{3}\vec{i} + \vec{j}$ and $\vec{i} - \sqrt{3}\vec{j}$ are perpendicular.
Since $3\vec{i} + \sqrt{3}\vec{j}$ and $\sqrt{3}\vec{i} + \vec{j}$ are parallel, $3\vec{i} + \sqrt{3}\vec{j}$ and $\vec{i} - \sqrt{3}\vec{j}$ are perpendicular, too.

18. In general, $\vec{u}$ and $\vec{v}$ are perpendicular when $\vec{u} \cdot \vec{v} = 0$.
In this case, $\vec{u} \cdot \vec{v} = (t\vec{i} - \vec{j} + \vec{k}) \cdot (t\vec{i} + t\vec{j} - 2\vec{k}) = t^2 - t - 2$.
This is zero when $t^2 - t - 2 = 0$, i.e. when $(t-2)(t+1) = 0$, so $t = 2$ or -1.
In general, $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{v} = \alpha\vec{u}$ for some real number α.
Thus we need $\alpha t\vec{i} - \alpha\vec{j} + \alpha\vec{k} = t\vec{i} + t\vec{j} - 2\vec{k}$, so we need $\alpha t = t$, and $-\alpha = t$, and $\alpha = -2$. But if $\alpha = -2$, we can't have $\alpha t = t$ unless $t = 0$, and if $t = 0$, we can't have $-\alpha = t$, so there are no values of t for which $\vec{u}$ and $\vec{v}$ are parallel.

19. Vectors $\vec{v}_1$, $\vec{v}_4$, and $\vec{v}_8$ are all parallel to each other. Vectors $\vec{v}_3$, $\vec{v}_5$, and $\vec{v}_7$ are all parallel to each other, and are all perpendicular to the vectors in the previous sentence. Vectors $\vec{v}_2$ and $\vec{v}_9$ are perpendicular.

20. (a) Perpendicular vectors have a dot product of 0. Since $\vec{a} \cdot \vec{c} = 1(-2) - 3(-1) - 1 \cdot 1 = 0$, and $\vec{b} \cdot \vec{d} = 1(-1) + 1(-1) + 2 \cdot 1 = 0$, the pairs we want are $\vec{a}, \vec{c}$ and $\vec{b}, \vec{d}$.
(b) Parallel vectors are multiples of one another, so there are no parallel vectors in this set.
(c) Since $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}||\cos\theta$, the dot product of the vectors we want is positive. We have

$$\vec{a} \cdot \vec{b} = 1 \cdot 1 - 3 \cdot 1 - 1 \cdot 2 = -4$$
$$\vec{a} \cdot \vec{d} = 1(-1) - 3(-1) - 1 \cdot 1 = 1$$
$$\vec{b} \cdot \vec{c} = 1(-2) + 3(-1) + 2 \cdot 1 = -1$$
$$\vec{c} \cdot \vec{d} = -2(-1) - 1(-1) + 1 \cdot 1 = 4,$$

and we already know $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{d} = 0$. Thus, the pairs of vectors with an angle of less than $\pi/2$ between them are $\vec{a}, \vec{d}$ and $\vec{c}, \vec{d}$.
(d) Vectors with an angle of more than $\pi/2$ between them have a negative dot product, so pairs are $\vec{a}, \vec{b}$ and $\vec{b}, \vec{c}$.

21. Let

$$\vec{a} = \vec{a}_{\text{parallel}} + \vec{a}_{\text{perp}}$$

where $\vec{a}_{\text{parallel}}$ is parallel to $\vec{d}$, and $\vec{a}_{\text{perp}}$ is perpendicular to $\vec{d}$. Then $\vec{a}_{\text{parallel}}$ is the projection of $\vec{a}$ in the direction of $\vec{d}$:

$$\begin{aligned}
\vec{a}_{\text{parallel}} &= \left(\vec{a} \cdot \frac{\vec{d}}{\|\vec{d}\|}\right) \frac{\vec{d}}{\|\vec{d}\|} \\
&= \left((3\vec{i} + 2\vec{j} - 6\vec{k}) \cdot \frac{(2\vec{i} - 4\vec{j} + \vec{k})}{\sqrt{2^2 + 4^2 + 1^2}}\right) \frac{(2\vec{i} - 4\vec{j} + \vec{k})}{\sqrt{2^2 + 4^2 + 1^2}} \\
&= -\frac{8}{21}(2\vec{i} - 4\vec{j} + \vec{k}) \\
&= -\frac{8}{21}\vec{d}
\end{aligned}$$

Since we now know $\vec{a}$ and $\vec{a}_{\text{parallel}}$, we can solve for $\vec{a}_{\text{perp}}$:

$$\begin{aligned}
\vec{a}_{\text{perp}} &= \vec{a} - \vec{a}_{\text{parallel}} \\
&= (3\vec{i} + 2\vec{j} - 6\vec{k}) - \left(-\frac{8}{21}\right)(2\vec{i} - 4\vec{j} + \vec{k}) \\
&= \frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k}.
\end{aligned}$$

Thus we can now write $\vec{a}$ as the sum of two vectors, one parallel to $\vec{d}$, the other perpendicular to $\vec{d}$:

$$\vec{a} = -\frac{8}{21}\vec{d} + \left(\frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k}\right)$$

22. Since a normal vector of the plane is $\vec{n} = -\vec{i} + 2\vec{j} + \vec{k}$, an equation for the plane is

$$\begin{aligned}
-x + 2y + z &= -1 + 2 \cdot 0 + 2 = 1 \\
-x + 2y + z &= 1.
\end{aligned}$$

23. Since the plane is normal to the vector $5\vec{i} + \vec{j} - 2\vec{k}$ and passes through the point $(0, 1, -1)$, an equation for the plane is

$$\begin{aligned}
5x + y - 2z &= 5 \cdot 0 + 1 \cdot 1 + (-2) \cdot (-1) = 3 \\
5x + y - 2z &= 3.
\end{aligned}$$

24. Since the plane is normal to the vector $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passes through the point $(1, -1, 2)$, an equation for the plane is

$$\begin{aligned}
2x - 3y + 7z &= 2 \cdot 1 - 3 \cdot (-1) + 7 \cdot 2 = 19 \\
2x - 3y + 7z &= 19.
\end{aligned}$$

25. Two planes are parallel if their normal vectors are parallel. Since the plane $2x + 4y - 3z = 1$ has normal vector $\vec{n} = 2\vec{i} + 4\vec{j} - 3\vec{k}$, the plane we are looking for has the same normal vector and passes through the point $(1, 0, -1)$. Thus the plane we want has equation:

$$2x + 4y - 3z = 2 \cdot 1 + 4 \cdot 0 + (-3) \cdot (-1) = 5$$

26. Two planes are parallel if their normal vectors are parallel. Since the plane $3x + y + z = 4$ has normal vector $\vec{n} = 3\vec{i} + \vec{j} + \vec{k}$, the plane we are looking for has the same normal vector and passes through the point $(-2, 3, 2)$. Thus, it has the equation

$$3x + y + z = 3 \cdot (-2) + 3 + 2 = -1.$$

27. (a) The plane can be written as $5x - 2y - z + 7 = 0$, so the vector $5\vec{i} - 2\vec{j} - \vec{k}$ is normal to the plane. The vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ is parallel to $5\vec{i} - 2\vec{j} - \vec{k}$ if one is a scalar multiple of the other. This occurs if the coefficients are in proportion:

$$\frac{\lambda}{5} = \frac{1}{-2} = \frac{0.5}{-1}.$$

Solving gives $\lambda = -2.5$.

(b) Substituting $x = a + 1$, $y = a$, $z = a - 1$ into the equation of the plane gives

$$\begin{aligned} a - 1 &= 5(a+1) - 2a + 7 \\ a - 1 &= 5a + 5 - 2a + 7 \\ -13 &= 2a \\ a &= -6.5. \end{aligned}$$

28. The plane cuts the x-axis where $y = z = 0$, so $x = -3/5 = -0.6$, giving the point

$$P = (-0.6, 0, 0)$$

Similarly, the plane cuts the y-axis where $x = z = 0$, so $y = 3/4 = 0.75$, so

$$Q = (0, 0.75, 0)$$

The plane cuts the z-axis at $x = y = 0$, so that $z = 3$, so

$$R = (0, 0, 3)$$

Now we have the three vertices of the triangle, P, Q, and R. The vectors along the three sides of the triangle are

$$\begin{aligned} \overrightarrow{QP} &= -0.6\vec{i} - 0.75\vec{j} \\ \overrightarrow{RP} &= -0.6\vec{i} - 3\vec{k} \\ \overrightarrow{QR} &= -0.75\vec{j} + 3\vec{k} \end{aligned}$$

The lengths of the sides of the triangle are

$$\begin{aligned} \|\overrightarrow{QP}\| &= \sqrt{(-0.6)^2 + (-0.75)^2} \approx 0.96 \\ \|\overrightarrow{RP}\| &= \sqrt{(-0.6)^2 + (-3)^2} \approx 3.059 \\ \|\overrightarrow{QR}\| &= \sqrt{(-0.75)^2 + (3)^2} \approx 3.092 \end{aligned}$$

The angle between the vectors $\vec{v}$ and $\vec{w}$ is given by

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} \quad \text{so} \quad \theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Thus,

$$\begin{aligned} \text{Angle at } P &= \arccos\left(\frac{\overrightarrow{QP} \cdot \overrightarrow{RP}}{\|\overrightarrow{QP}\|\|\overrightarrow{RP}\|}\right) \\ &\approx \arccos\left(\frac{(-0.6\vec{i} - 0.75\vec{j}) \cdot (-0.6\vec{i} - 3\vec{k})}{0.96 \cdot 3.059}\right) \approx \arccos\left(\frac{0.36}{0.96 \cdot 3.059}\right) \\ &\approx \arccos(0.123) \approx 83.0^\circ. \end{aligned}$$

$$\begin{aligned} \text{Angle at } Q &= \arccos\left(\frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\|\overrightarrow{QP}\|\|\overrightarrow{QR}\|}\right) \\ &\approx \arccos\left(\frac{(-0.6\vec{i} - 0.75\vec{j}) \cdot (-0.75\vec{j} + 3\vec{k})}{0.96 \cdot 3.092}\right) \approx \arccos\left(\frac{0.5625}{0.96 \cdot 3.092}\right) \\ &\approx \arccos(0.19) \approx 79.1^\circ. \end{aligned}$$

Now we use the fact that the angles of the triangle add up to 180°. Thus

$$\text{Angle at } R \approx 180^\circ - (83.0^\circ + 79.1^\circ) \approx 17.9^\circ.$$

29. The angle between two planes is equal to the angle between the normal vectors of the two planes. A normal vector to the plane $5(x-1)+3(y+2)+2z=0$ is

$$\vec{n}_1 = 5\vec{i} + 3\vec{j} + 2\vec{k},$$

and a normal vector to the plane $x+3(y-1)+2(z+4)=0$ is

$$\vec{n}_2 = \vec{i} + 3\vec{j} + 2\vec{k}.$$

Since $\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\|\|\vec{n}_2\|\cos\theta$, then

$$\begin{aligned}\cos\theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{(5\vec{i} + 3\vec{j} + 2\vec{k}) \cdot (\vec{i} + 3\vec{j} + 2\vec{k})}{\sqrt{5^2+3^2+2^2}\sqrt{1^2+3^2+2^2}} \\ &= \frac{18}{\sqrt{532}} = 0.78\end{aligned}$$

Hence, $\theta \approx 38.7^\circ$.

30. We first find displacement vectors $\overrightarrow{AB} = (4-2)\vec{i} + (2-2)\vec{j} + (1-2)\vec{k} = 2\vec{i} - \vec{k}$ and $\overrightarrow{AC} = (2-2)\vec{i} + (3-2)\vec{j} + (1-2)\vec{k} = \vec{j} - \vec{k}$. Then

$$\begin{aligned}\cos(\angle BAC) &= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} \\ &= \frac{1}{\sqrt{5}\sqrt{2}} \\ &= 0.3162.\end{aligned}$$

Thus angle BAC is 71.57° (or 1.25 radians.)

31. See Figure 13.21. One way to find the angle at A is to find the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Since $\overrightarrow{AB} = -1\vec{i} - 7\vec{j}$ and $\overrightarrow{AC} = -5\vec{i} - 3\vec{j}$, we have

$$\begin{aligned}\cos(\angle BAC) &= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} \\ &= \frac{(-1)(-5)+(-7)(-3)}{\sqrt{50}\sqrt{34}} \\ &= 0.6306.\end{aligned}$$

Thus the angle at vertex A is 50.91°. Similarly, we see that the angle at vertex B is 53.13° and (since the angles of a triangle add up to 180°) the angle at vertex C is 75.96°.

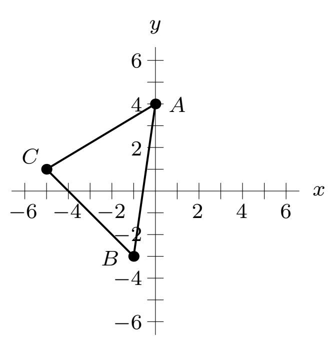

Figure 13.21

32. (a) The points A, B and C are shown in Figure 13.22.

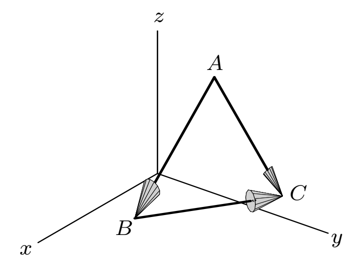

Figure 13.22

First, we calculate the vectors which form the sides of this triangle:
$\overrightarrow{AB} = (4\vec{i} + 2\vec{j} + \vec{k}) - (2\vec{i} + 2\vec{j} + 2\vec{k}) = 2\vec{i} - \vec{k}$
$\overrightarrow{BC} = (2\vec{i} + 3\vec{j} + \vec{k}) - (4\vec{i} + 2\vec{j} + \vec{k}) = -2\vec{i} + \vec{j}$
$\overrightarrow{AC} = (2\vec{i} + 3\vec{j} + \vec{k}) - (2\vec{i} + 2\vec{j} + 2\vec{k}) = \vec{j} - \vec{k}$

Now we calculate the lengths of each of the sides of the triangles:
$\|\overrightarrow{AB}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$
$\|\overrightarrow{BC}\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$
$\|\overrightarrow{AC}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
Thus the length of the shortest side of S is $\sqrt{2}$.

(b) $\cos \angle BAC = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|} = \frac{2 \cdot 0 + 0 \cdot 1 + (-1) \cdot (-1)}{\sqrt{5} \cdot \sqrt{2}} \approx 0.32$

33. We need to find the speed of the wind in the direction of the track. Looking at Figure 13.23, we see that we want the component of $\vec{w}$ in the direction of $\vec{v}$. We calculate

$$\|\vec{w}_{\text{parallel}}\| = \|\vec{w}\| \cos\theta = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|} = \frac{(5\vec{i} + \vec{j}) \cdot (2\vec{i} + 6\vec{j})}{\|2\vec{i} + 6\vec{j}\|}$$
$$= \frac{16}{\sqrt{40}} \approx 2.53$$
$$< 5$$

Therefore, the race results will not be disqualified.

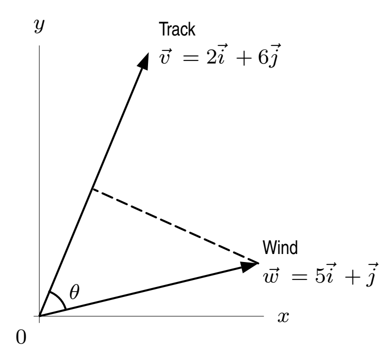

Figure 13.23

34. Let the room be put in the coordinate system as shown in Figure 13.24.

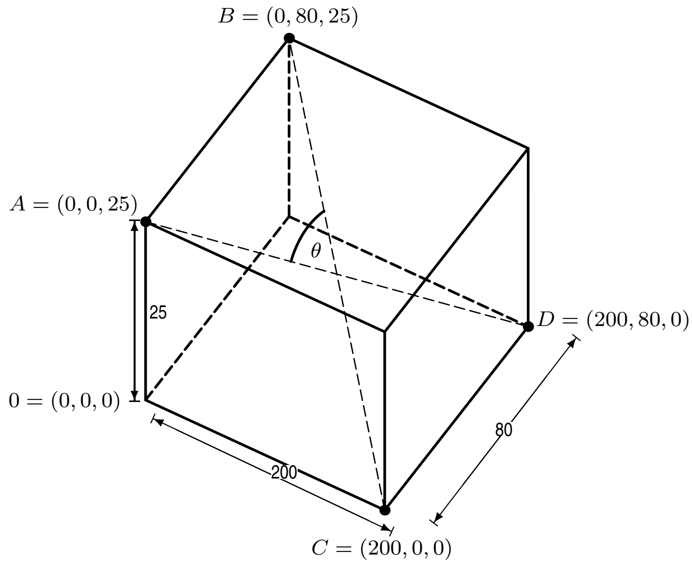

Figure 13.24

Then the vectors of the two strings are given by:
$\overrightarrow{AD} = (200\vec{i} + 80\vec{j} + 0\vec{k}) - (0\vec{i} + 0\vec{j} + 25\vec{k}) = 200\vec{i} + 80\vec{j} - 25\vec{k}$
$\overrightarrow{BC} = (200\vec{i} + 0\vec{j} + 0\vec{k}) - (0\vec{i} + 80\vec{j} + 25\vec{k}) = 200\vec{i} - 80\vec{j} - 25\vec{k}$.
Let the angle between $\overrightarrow{AD}$ and $\overrightarrow{BC}$ be θ. Then we have

$$\begin{aligned}\cos\theta &= \frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{\|\overrightarrow{AD}\|\,\|\overrightarrow{BC}\|}\\ &= \frac{200(200)+(80)(-80)+(-25)(-25)}{\sqrt{200^2+80^2+(-25)^2}\sqrt{(200)^2+(-80)^2+(-25)^2}}\\ &= \frac{34225}{47025}\\ &= 0.727804\end{aligned}$$

35. The vector $\vec{a}$ represents the averages of the exams, written as decimals. The vector $\vec{w}$ represents the weightings.

$$\vec{w}\cdot\vec{a} = 0.1\cdot 0.75 + 0.15\cdot 0.91 + 0.25\cdot 0.84 + 0.5\cdot 0.87 = 0.8565 = 85.65\%$$

The dot product, 86.65%, represents the class average of the four exams in the course.

36. We have

$$\begin{aligned}\vec{p}\cdot\vec{q} &= (1.00)(43) + (3.50)(57) + (4.00)(12) + (2.75)(78) + (5.00)(20) + (3.00)(35)\\ &= 710 \text{ dollars.}\end{aligned}$$

The vendor took in \$710 in from sales. The quantity $\vec{p}\cdot\vec{q}$ represents the total revenue earned.

37. If $\vec{x}$ and $\vec{y}$ are two consumption vectors corresponding to points satisfying the same budget constraint, then

$$\vec{p}\cdot\vec{x} = k = \vec{p}\cdot\vec{y}.$$

Therefore we have

$$\vec{p}\cdot(\vec{x}-\vec{y}) = \vec{p}\cdot\vec{x} - \vec{p}\cdot\vec{y} = 0.$$

Thus $\vec{p}$ and $\vec{x}-\vec{y}$ are perpendicular; that is, the difference between two consumption vectors on the same budget constraint is perpendicular to the price vector.

38. (a) The geometric definition of the dot product says that

$$\vec{n}\cdot\overrightarrow{P_0P}=||\vec{n}||||\overrightarrow{P_0P}||\cos\theta,$$

where θ is the angle between $\vec{n}$ and $\overrightarrow{P_0P}$ with $0\le\theta\le\pi$. To say that the dot product $\vec{n}\cdot\overrightarrow{P_0P}$ is positive means that the angle between $\vec{n}$ and $\overrightarrow{P_0P}$ is between 0 and $\pi/2$, and strictly less than $\pi/2$. Hence $\vec{n}$ and $\overrightarrow{P_0P}$ are both pointing to the same side of the plane. Thus, all the points satisfying $\vec{n}\cdot\overrightarrow{P_0P}>0$ are on the same side of the plane, the side which $\vec{n}$ points to. To say that the dot product is negative is to say that $\pi/2<\theta\le\pi$, and this means that $\overrightarrow{P_0P}$ and $\vec{n}$ are pointing to opposite sides of the plane. Thus, all points satisfying $\vec{n}\cdot\overrightarrow{P_0P}<0$ are on the side of the plane opposite to $\vec{n}$.

(b) Suppose the normal vector is $\vec{n}=a\vec{i}+b\vec{j}+c\vec{k}$, let $P_0=(x_0,y_0,z_0)$ be a point in the plane and let $P=(x,y,z)$ be a variable point. Then $\overrightarrow{P_0P}=(x-x_0)\vec{i}+(y-y_0)\vec{j}+(z-z_0)\vec{k}$. Then $\vec{n}\cdot\overrightarrow{P_0P}>0$ means

$$a(x-x_0)+b(y-y_0)+c(z-z_0)>0$$

and $\vec{n}\cdot\overrightarrow{P_0P}<0$ means

$$a(x-x_0)+b(y-y_0)+c(z-z_0)<0$$

If the equation of the plane is written $ax+by+cz=d$ (with $d=ax_0+by_0+cz_0$) then the inequalities become

$$ax+by+cz>d \quad\text{and}\quad ax+by+cz<d.$$

(c) We test each of the points $P=(-1,-1,1)$, $Q=(-1,-1,-1)$ and $R=(1,1,1)$, using the coordinate version of the inequalities in part (b):

$$\begin{aligned}
P:&\quad 2\cdot(-1)-3\cdot(-1)+4\cdot 1=5>4\\
Q:&\quad 2\cdot(-1)-3\cdot(-1)+4\cdot(-1)=-3<4\\
R:&\quad 2\cdot 1-3\cdot 1+4\cdot 1=3<4
\end{aligned}$$

Therefore Q and R are on the same side of the plane as each other; P is on the other side.

39. Suppose $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $\vec{w}=w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$.

- Property 1:

  We calculate both $\vec{v}\cdot\vec{w}$ and $\vec{w}\cdot\vec{v}$ using the algebraic definition of the dot product:

$$\vec{v}\cdot\vec{w}=v_1w_1+v_2w_2+v_3w_3$$

$$\vec{w}\cdot\vec{v}=w_1v_1+w_2v_2+w_3v_3$$

But since ordinary multiplication of scalars is commutative, $v_1w_1=w_1v_1$ and so on. Therefore

$$\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}.$$

- Property 2:

  First we observe that

$$\lambda\vec{w}=\lambda(w_1\vec{i}+w_2\vec{j}+w_3\vec{k})=(\lambda w_1)\vec{i}+(\lambda w_2)\vec{j}+(\lambda w_3)\vec{k}$$

$$\lambda\vec{v}=\lambda(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})=(\lambda v_1)\vec{i}+(\lambda v_2)\vec{j}+(\lambda v_3)\vec{k}.$$

Now we calculate the three quantities $\vec{v}\cdot(\lambda\vec{w})$ and $\lambda(\vec{v}\cdot\vec{w})$ and $(\lambda\vec{v})\cdot\vec{w}$

$$\begin{aligned}
\vec{v}\cdot(\lambda\vec{w})&=v_1(\lambda w_1)+v_2(\lambda w_2)+v_3(\lambda w_3)\\
\lambda(\vec{v}\cdot\vec{w})&=\lambda(v_1w_1+v_2w_2+v_3w_3)\\
(\lambda\vec{v})\cdot\vec{w}&=(\lambda v_1)w_1+(\lambda v_2)w_2+(\lambda v_3)w_3
\end{aligned}$$

Since ordinary multiplication is associative and commutative, we know that $v_1(\lambda w_1)=\lambda v_1w_1=(\lambda v_1)w_1$ and so on. Thus, we have $\vec{v}\cdot(\lambda\vec{w})=(\lambda\vec{v})\cdot\vec{w}$.

In addition, the distributive property of ordinary multiplication tells us that

$$\lambda(v_1w_1+v_2w_2+v_3w_3)=\lambda v_1w_1+\lambda v_2w_2+\lambda v_3w_3$$

Thus, we know that all three quantities are equal

$$\vec{v}\cdot(\lambda\vec{w})=\lambda(\vec{v}\cdot\vec{w})=(\lambda\vec{v})\cdot\vec{w}$$

- Property 3:

  First we observe that

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{i} + (v_2 + w_2)\vec{j} + (v_3 + w_3)\vec{k}.$$

Next we calculate the quantities $((\vec{v} + \vec{w}) \cdot \vec{u})$ and $(\vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u})$.

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot \vec{u} &= (v_1 + w_1)u_1 + (v_2 + w_2)u_2 + (v_3 + w_3)u_3 \\ \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} &= (v_1u_1 + v_2u_2 + v_3u_3) + (w_1u_1 + w_2u_2 + w_3u_3).\end{aligned}$$

The distributive law of ordinary multiplication shows that $(v_1 + w_1)u_1 = v_1u_1 + w_1u_1$, and so on. Thus, the dot product is distributive also:

$$(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$$

40. Property 2 says that multiplying one of the vectors by a scalar simply multiplies the dot product by the same scalar. If $\lambda > 0$, then when one vector is multiplied by λ, the angle between the vectors does not change, but the length of one vector, and hence the dot product, is multiplied by λ. The result remains true when $\lambda < 0$. For a justification in the case when $\lambda < 0$, see Problem 44 on page 926.

41. We want to show that $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and $\vec{c}$ are perpendicular. We do this by taking their dot product:

$$((\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}) \cdot \vec{c} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{c}) = 0.$$

Since the dot product is 0, the vectors $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and $\vec{c}$ are perpendicular.

42. Since $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$, $(\vec{u} - \vec{v}) \cdot \vec{w} = 0$. This equality holds for any $\vec{w}$, so we can take $\vec{w} = \vec{u} - \vec{v}$. This gives

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = 0,$$

that is,

$$\|\vec{u} - \vec{v}\| = 0.$$

This implies $\vec{u} - \vec{v} = 0$, that is, $\vec{u} = \vec{v}$.

43. If $\vec{u} = \vec{0}$, then both sides of the equation are zero. If $\vec{u} \neq \vec{0}$, write $\vec{v}_{\text{parallel}}$, $\vec{w}_{\text{parallel}}$, and $(\vec{v} + \vec{w})_{\text{parallel}}$ for the components of $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ in the direction of $\vec{u}$. Then Figure 13.36 shows that

$$\vec{v}_{\text{parallel}} + \vec{w}_{\text{parallel}} = (\vec{v} + \vec{w})_{\text{parallel}}.$$

So

$$\left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} + \left(\frac{\vec{w} \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{(\vec{v} + \vec{w}) \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u}.$$

Thus, since $\vec{u} \neq \vec{0}$, we deduce that

$$\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} + \frac{\vec{w} \cdot \vec{u}}{\|\vec{u}\|^2} - \frac{(\vec{v} + \vec{w}) \cdot \vec{u}}{\|\vec{u}\|^2} = 0,$$

so

$$\vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = (\vec{v} + \vec{w}) \cdot \vec{u}.$$

44. Suppose θ is the angle between $\vec{u}$ and $\vec{v}$.

(a) By the definition of scalar multiplication, we know that $-\vec{v}$ is in the opposite direction of $\vec{v}$, so the angle between $\vec{u}$ and $-\vec{v}$ is $\pi - \theta$. (See Figure 13.25.) Hence,

$$\begin{aligned}\vec{u} \cdot (-\vec{v}) &= \|\vec{u}\|\|-\vec{v}\|\cos(\pi - \theta) \\ &= \|\vec{u}\|\|\vec{v}\|(-\cos\theta) \\ &= -(\vec{u} \cdot \vec{v})\end{aligned}$$

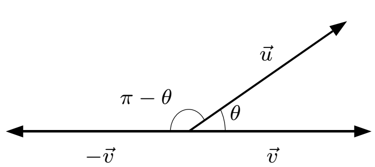

Figure 13.25

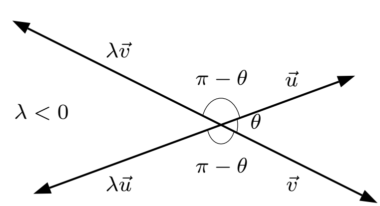

Figure 13.26

(b) If $\lambda < 0$, the angle between $\vec{u}$ and $\lambda\vec{v}$ is $\pi - \theta$, and so is the angle between $\lambda\vec{u}$ and $\vec{v}$. (See Figure 13.26.) So we have,

$$\begin{aligned}
\vec{u}\cdot(\lambda\vec{v}) &= \|\vec{u}\|\|\lambda\vec{v}\|\cos(\pi-\theta)\\
&= |\lambda|\|\vec{u}\|\|\vec{v}\|(-\cos\theta)\\
&= -\lambda\|\vec{u}\|\|\vec{v}\|(-\cos\theta) \quad \text{since } |\lambda| = -\lambda\\
&= \lambda\|\vec{u}\|\|\vec{v}\|\cos\theta\\
&= \lambda(\vec{u}\cdot\vec{v})
\end{aligned}$$

By a similar argument, we have

$$\begin{aligned}
(\lambda\vec{u})\cdot\vec{v} &= \|\lambda\vec{u}\|\|\vec{v}\|\cos(\pi-\theta)\\
&= -\lambda\|\vec{u}\|\|\vec{v}\|(-\cos\theta)\\
&= \lambda(\vec{u}\cdot\vec{v})
\end{aligned}$$

45. Let $\vec{u}$ and $\vec{v}$ be the displacement vectors from C to the other two vertices. Then

$$\begin{aligned}
c^2 &= \|\vec{u}-\vec{v}\|^2\\
&= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\
&= \vec{u}\cdot\vec{u}-\vec{v}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}\\
&= \|\vec{u}\|^2 - 2\|u\|\|v\|\cos C + \|\vec{v}\|^2\\
&= a^2 - 2ab\cos C + b^2
\end{aligned}$$

46. We substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and by the result of Problem 43, we expand as follows:

$$\begin{aligned}
(\vec{u}\cdot\vec{v})_{\text{geom}} &= (u_1\vec{i}+u_2\vec{j}+u_3\vec{k})\cdot\vec{v}\\
&= (u_1\vec{i})\cdot\vec{v}+(u_2\vec{j})\cdot\vec{v}+(u_3\vec{k})\cdot\vec{v}
\end{aligned}$$

where all the dot products are defined geometrically By the result of Problem 44 we can write

$$(\vec{u}\cdot\vec{v})_{\text{geom}} = u_1(\vec{i}\cdot\vec{v})_{\text{geom}} + u_2(\vec{j}\cdot\vec{v})_{\text{geom}} + u_3(\vec{k}\cdot\vec{v})_{\text{geom}}.$$

Now substitute $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and expand, again using Problem 43 and the geometric definition of the dot product:

$$\begin{aligned}
(\vec{u}\cdot\vec{v})_{\text{geom}} &= u_1\left(\vec{i}\cdot(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})\right)_{\text{geom}}\\
&\quad +u_2\left(\vec{j}\cdot(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})\right)_{\text{geom}}\\
&\quad +u_3\left(\vec{k}\cdot(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})\right)_{\text{geom}}\\
&= u_1v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_1v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_1v_3(\vec{i}\cdot\vec{k})_{\text{geom}}\\
&\quad +u_2v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_2v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_2v_3(\vec{i}\cdot\vec{k})_{\text{geom}}\\
&\quad +u_3v_1(\vec{i}\cdot\vec{i})_{\text{geom}} + u_3v_2(\vec{i}\cdot\vec{j})_{\text{geom}} + u_3v_3(\vec{i}\cdot\vec{k})_{\text{geom}}
\end{aligned}$$

The geometric definition of the dot product shows that

$$\begin{aligned}
\vec{i}\cdot\vec{i} &= \|\vec{i}\|\,\|\vec{i}\|\cos 0 = 1\\
\vec{i}\cdot\vec{j} &= \|\vec{i}\|\,\|\vec{j}\|\cos\frac{\pi}{2} = 0.
\end{aligned}$$

Similarly $\vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ and $\vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$. Thus, the expression for $(\vec{u}\cdot\vec{v})_{\text{geom}}$ becomes

$$\begin{aligned}
(\vec{u}\cdot\vec{v})_{\text{geom}} &= u_1v_1(1)+u_1v_2(0)+u_1v_3(0)\\
&\quad +u_2v_1(0)+u_2v_2(1)+u_2v_3(0)\\
&\quad +u_3v_1(0)+u_3v_2(0)+u_3v_3(1)\\
&= u_1v_1+u_2v_2+u_3v_3.
\end{aligned}$$

47. (a) Since $q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}) = \|\vec{v} + t\vec{w}\|^2$ and since the length of any vector is nonnegative, we must have

$$q(t) = \|\vec{v} + t\vec{w}\|^2 \geq 0$$

for all real t.

(b) Using the distributive law

$$\begin{aligned} q(t) &= (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}) = \vec{v} \cdot \vec{v} + t\vec{w} \cdot \vec{v} + \vec{v} \cdot t\vec{w} + t^2\vec{w} \cdot \vec{w} \\ &= \|\vec{v}\|^2 + 2(\vec{v} \cdot \vec{w})t + \|\vec{w}\|^2 t^2. \end{aligned}$$

If $\vec{w} \neq 0$, then $\|\vec{w}\| \neq 0$ and $q(t)$ is quadratic in t.

(c) Since $q(t) \geq 0$, the quadratic has one repeated root or no roots, so the discriminant must be less than or equal to zero. Thus,

$$(2\vec{v} \cdot \vec{w})^2 - 4\|\vec{v}\|^2\|\vec{w}\|^2 \leq 0.$$

Taking square roots, we have

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

If $\vec{w} = 0$, then $q(t)$ is no longer a quadratic. However, in that case,

$$|\vec{v} \cdot \vec{w}| = 0 = \|\vec{v}\|\|\vec{w}\|$$

so the inequality still holds.

Solutions for Section 13.4

Exercises

1. $\vec{v} \times \vec{w} = \vec{k} \times \vec{j} = -\vec{i}$ (remember $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the axes, and you must use the right hand rule.)

2. $\vec{v} = -\vec{i}$, and $\vec{w} = \vec{j} + \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{j} - \vec{k}$$

3. $\vec{v} = \vec{i} + \vec{k}$, and $\vec{w} = \vec{i} + \vec{j}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} + \vec{k}$$

4. $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, and $\vec{w} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\vec{i} + 2\vec{j}$$

5. $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$, and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 3\vec{j} + 7\vec{k}$$

6. $\vec{v} = 2\vec{i} - \vec{j} - \vec{k}$, and $\vec{w} = -6\vec{i} + 3\vec{j} + 3\vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ -6 & 3 & 3 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}.$$

7.

$$\begin{aligned}[(\vec{i}+\vec{j})\times\vec{i}]\times\vec{j} &= (\vec{i}\times\vec{i}+\vec{j}\times\vec{i})\times\vec{j}\\ &= (\vec{0}-\vec{k})\times\vec{j}\\ &= -\vec{k}\times\vec{j}\\ &= \vec{j}\times\vec{k}=\vec{i}.\end{aligned}$$

8.

$$\begin{aligned}(\vec{i}+\vec{j})\times(\vec{i}\times\vec{j}) &= (\vec{i}+\vec{j})\times\vec{k}\\ &= (\vec{i}\times\vec{k})+(\vec{j}\times\vec{k})\\ &= -\vec{j}+\vec{i}=\vec{i}-\vec{j}.\end{aligned}$$

9. By the definition of cross product, $2\vec{i}\times(\vec{i}+\vec{j})$ is in the direction of $\vec{k}$. The magnitude of it equals to the area of the parallelogram which is

$$\|2\vec{i}\|\cdot\|\vec{i}+\vec{j}\|\sin\frac{\pi}{4}=2\sqrt{2}\sin\frac{\pi}{4}=2\sqrt{2}\cdot\frac{\sqrt{2}}{2}=2.$$

So $2\vec{i}\times(\vec{i}+\vec{j})=2\vec{k}$. See Figure 13.27.

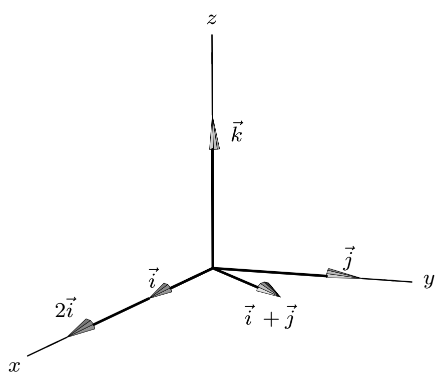

Figure 13.27

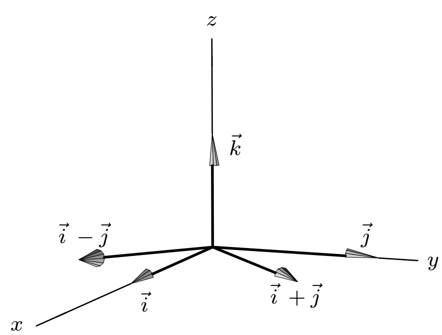

Figure 13.28

10. By definition, $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})$ is in the direction of $-\vec{k}$. The magnitude is

$$\|\vec{i}+\vec{j}\|\cdot\|\vec{i}-\vec{j}\|\sin\frac{\pi}{4}=\sqrt{2}\cdot\sqrt{2}=2.$$

So $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})=-2\vec{k}$. See Figure 13.28.

11. We find that $\vec{v}\times\vec{w}=-6\vec{i}+7\vec{j}+8\vec{k}$ and $\vec{w}\times\vec{v}=6\vec{i}-7\vec{j}-8\vec{k}$. Notice that

$$\vec{v}\times\vec{w}=-(\vec{w}\times\vec{v}).$$

12.

$$\begin{aligned}\vec{a}\times\vec{b} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & -1\\ 1 & -4 & 2\end{vmatrix}\\ &= \begin{vmatrix}1 & -1\\ -4 & 2\end{vmatrix}\vec{i}-\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix}\vec{j}+\begin{vmatrix}3 & 1\\ 1 & -4\end{vmatrix}\vec{k}\\ &= -2\vec{i}-7\vec{j}-13\vec{k}.\end{aligned}$$

Since

$$\vec a \cdot (\vec a \times \vec b\,) = 3(-2) + (-7) - (-13) = 0$$

and

$$\vec b \cdot (\vec a \times \vec b\,) = 1(-2) - 4(-7) + 2(-13) = 0,$$

$\vec a \times \vec b$ is perpendicular to both $\vec a$ and $\vec b$.

13. We can form the displacement vectors $\vec a = -\vec i + \vec j + 0\vec k$ from $(1,0,0)$ to $(0,1,0)$ and $\vec b = -\vec i + 0\vec j + \vec k$ from $(1,0,0)$ to $(0,0,1)$. A normal vector to the plane is $\vec a \times \vec b = \vec i + \vec j + \vec k$. Using the point $(1,0,0)$, the plane can be written as $(x-1)+y+z=0$ or $x+y+z=1$.

14. The displacement vector from $(3,4,2)$ to $(-2,1,0)$ is:

$$\vec a = -5\vec i - 3\vec j - 2\vec k\,.$$

The displacement vector from $(3,4,2)$ to $(0,2,1)$ is:

$$\vec b = -3\vec i - 2\vec j - \vec k\,.$$

Therefore the vector normal to the plane is:

$$\vec n = \vec a \times \vec b = -\vec i + \vec j + \vec k\,.$$

Using the first point, the equation of the plane can be written as:

$$-(x-3) + (y-4) + (z-2) = 0.$$

The equation of the plane is thus:

$$-x + y + z = 3.$$

Problems

15. (a) Since $\vec v \cdot \vec w = ||\vec v\,||\,||\vec w\,|| \cos\theta$ and $||\vec v \times \vec w\,|| = ||\vec v\,||\,||\vec w\,|| \sin\theta$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{||\vec v \times \vec w\,||}{\vec v \cdot \vec w} = \frac{3}{5} = 0.6.$$

(b) Then $\theta = \tan^{-1}(0.6) = 0.540$.

16. Since

$$\|\vec v \times \vec w\,\| = \|\vec v\,\| \cdot \|\vec w\,\| \sin\theta,$$

and

$$\vec v \cdot \vec w = \|\vec v\,\| \cdot \|\vec w\,\| \cos\theta,$$

so

$$\frac{\|\vec v \times \vec w\,\|}{\vec v \cdot \vec w} = \frac{\|\vec v\,\| \cdot \|\vec w\,\| \sin\theta}{\|\vec v\,\| \cdot \|\vec w\,\| \cos\theta} = \tan\theta,$$

so

$$\tan\theta = \frac{\|2\vec i - 3\vec j + 5\vec k\,\|}{3}\frac{\sqrt{38}}{3} \approx 2.05.$$

17. Since $\vec v \times \vec w$ is perpendicular to both $\vec v$ and $\vec w$, we can conclude that $\vec v \times \vec w$ is parallel to the z-axis.

18. (a) We first find two displacement vectors: $\overrightarrow{AB} = (3-(-1))\vec i + (2-3)\vec j + (4-0)\vec k = 4\vec i - \vec j + 4\vec k$ and $\overrightarrow{AC} = 2\vec i - 4\vec j + 5\vec k$. The normal vector, $\vec n$, to the plane is perpendicular to these two vectors, so we have

$$\vec n = \overrightarrow{AB} \times \overrightarrow{AC} = 11\vec i - 12\vec j - 14\vec k\,.$$

Using the normal vector, we see that the equation of the plane is $11x - 12y - 14z = d$ for some number d. Substituting one of the points gives $d = -47$. Therefore, an equation for the plane is

$$11x - 12y - 14z = -47.$$

(b) The area of the triangle is given by

$$\text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{461} = 10.74.$$

19. (a) If we let $\overrightarrow{PQ}$ in Figure 13.29 be the vector from point P to point Q and $\overrightarrow{PR}$ be the vector from P to R, then

$$\overrightarrow{PQ} = -\vec{i} + 2\vec{k}$$
$$\overrightarrow{PR} = 2\vec{i} - \vec{k},$$

then the area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is:

$$\begin{array}{c}\text{Area of}\\ \text{parallelogram}\end{array} = \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} \right\| = \|3\vec{j}\| = 3.$$

Thus, the area of the triangle PQR is

$$\left(\begin{array}{c}\text{Area of}\\ \text{triangle}\end{array}\right) = \frac{1}{2}\left(\begin{array}{c}\text{Area of}\\ \text{parallelogram}\end{array}\right) = \frac{3}{2} = 1.5.$$

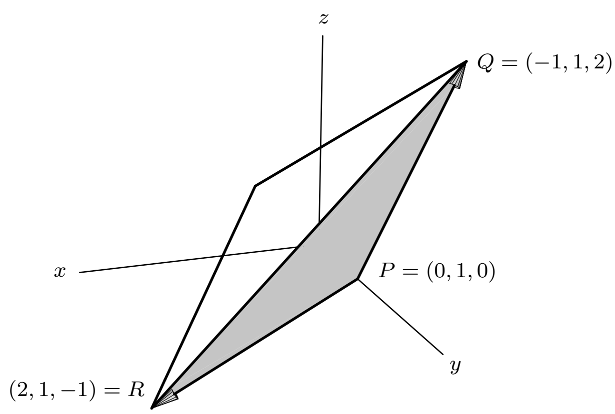

Figure 13.29

(b) Since $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane PQR, and from above, we have $\vec{n} = 3\vec{j}$, the equation of the plane has the form $3y = C$. At the point $(0, 1, 0)$ we get $3 = C$, therefore $3y = 3$, i.e., $y = 1$.

20. The normal vectors to the two planes are $\vec{n}_1 = 4\vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{n}_2 = \vec{i} + 5\vec{j} - \vec{k}$. A vector parallel to the line of intersection of the two planes is perpendicular to both these normal vectors, so

$$\text{Vector parallel to line} = \vec{n}_1 \times \vec{n}_2 = -7\vec{i} + 6\vec{j} + 23\vec{k}.$$

21. The normal vectors to the planes are $\vec{n_1} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{n_2} = 4\vec{i} + \vec{j} - 3\vec{k}$. The line of intersection is perpendicular to both normal vectors (picture the pages in a partially open book). Hence the vector we need is $\vec{n_1} \times \vec{n_2} = 4\vec{i} + 26\vec{j} + 14\vec{k}$.

22. The vector parallel to the line of intersection is $4\vec{i} + 26\vec{j} + 14\vec{k}$ and this is normal to the desired plane. Therefore, $4x + 26y + 14z = 0$ is the equation of the plane.

23. We use the same normal vector $\vec{n} = 4\vec{i} + 26\vec{j} + 14\vec{k}$ and the point $(4, 5, 6)$ to get $4(x-4) + 26(y-5) + 14(z-6) = 0$.

24. Normal vectors to the planes are

$$\vec{n}_1 = \vec{i} - \vec{j} + \vec{k} \quad \text{and} \quad \vec{n}_2 = 2\vec{i} + \vec{j} - 2\vec{k}.$$

The vector $\vec{n}_1 \times \vec{n}_2$ is perpendicular to both planes and is normal to the plane we want:

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \vec{i} + 4\vec{j} + 3\vec{k}.$$

The plane through the origin with normal $\vec{n}_1 \times \vec{n}_2$ is

$$x + 4y + 3z = 0.$$

25. First let

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} \quad \vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

so $\vec{b} + \vec{c} = (b_1 + c_1)\vec{i} + (b_2 + c_2)\vec{j} + (b_3 + c_3)\vec{k}$. Now, using the general formula for cross products, we have:

$$\begin{aligned}
&\vec{a} \times (\vec{b} + \vec{c}) \\
&= [a_2(b_3 + c_3) - a_3(b_2 + c_2)]\vec{i} + [a_3(b_1 + c_1) - a_1(b_3 + c_3)]\vec{j} + [a_1(b_2 + c_2) - a_2(b_1 + c_1)]\vec{k} \\
&= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\vec{i} + (a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3)\vec{j} \\
&\quad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\vec{k} \\
&= (a_2b_3 - a_3b_2)\vec{i} + (a_2c_3 - a_3c_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_3c_1 - a_1c_3)\vec{j} \\
&\quad + (a_1b_2 - a_2b_1)\vec{k} + (a_1c_2 - a_2c_1)\vec{k} \\
&= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} + (a_2c_3 - a_3c_2)\vec{i} + (a_3c_1 - a_1c_3)\vec{j} \\
&\quad + (a_1c_2 - a_2c_1)\vec{k} \\
&= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})
\end{aligned}$$

Thus, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

26. Any vector $\vec{v}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$ will have the property that its dot product with $\vec{a}$ and $\vec{b}$ is 0, that is

$$\begin{aligned}
\vec{a} \cdot \vec{v} &= a_1x + a_2y + a_3z = 0, \\
\vec{b} \cdot \vec{v} &= b_1x + b_2y + b_3z = 0.
\end{aligned}$$

Multiply the first equation by b_1 and the second by a_1 and subtract to get

$$(b_1a_2 - a_1b_2)y + (b_1a_3 - a_1b_3)z = 0 \quad \text{or} \quad y = \frac{-(b_1a_3 - a_1b_3)z}{(b_1a_2 - a_1b_2)} \quad (\text{for } b_1a_2 \neq a_1b_2)$$

Multiply the second equation by a_2 and the first by b_2 and subtract to get

$$(b_2a_1 - a_2b_1)x + (b_2a_3 - a_2b_3)z = 0 \quad \text{or} \quad x = \frac{-(b_2a_3 - a_2b_3)z}{(b_2a_1 - a_2b_1)}.$$

So

$$\vec{v} = \frac{-(b_2a_3 - a_2b_3)z}{(b_2a_1 - a_2b_1)}\vec{i} - \frac{(b_1a_3 - a_1b_3)z}{(b_1a_2 - a_1b_2)}\vec{j} + z\vec{k}.$$

Pick $z = b_2a_1 - b_1a_2$ and multiply out, and we see that the algebraic method of finding a cross product yields the same result as our standard method.

27. (a) Figure 13.30 shows the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfying the conditions $0 < a_2 < a_1$ and $0 < b_1 < b_2$.

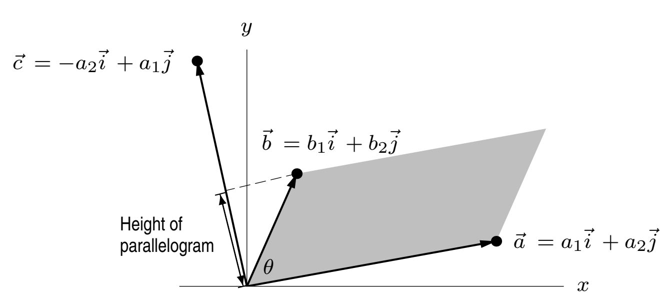

Figure 13.30

(b) $\vec{c} \cdot \vec{a} = -a_2a_1 + a_1a_2 = 0$ and $\vec{c} \cdot \vec{c} = {a_2}^2 + {a_1}^2 = \|\vec{a}\|^2$. Thus $\vec{c}$ is orthogonal (perpendicular) to $\vec{a}$ and has the same length as $\vec{a}$.

(c) $\vec{c} \cdot \vec{b} = -a_2b_1 + a_1b_2$. Since $a_1 > a_2 > 0$, and $b_2 > b_1 > 0$, we know that $\vec{c} \cdot \vec{b}$ is positive.

(d) If θ is the angle between $\vec{a}$ and $\vec{b}$ and α is the angle between $\vec{c}$ and $\vec{b}$, then $\alpha = (\frac{\pi}{2} - \theta)$. Thus $\cos\alpha = \sin\theta$, so

$$\vec{c} \cdot \vec{b} = \|\vec{c}\|\|\vec{b}\| \cos\alpha = \|\vec{a}\|\|\vec{b}\| \sin\theta.$$

Since $\|\vec{a}\| =$ Base of the parallelogram and
$\|\vec{b}\| \sin\theta =$ Height of the parallelogram, we have

$$\vec{c} \cdot \vec{b} = \text{Base} \cdot \text{Height} = \text{Area of the parallelogram formed by } \vec{a} \text{ and } \vec{b}.$$

(e) By the right-hand rule, $\vec{a} \times \vec{b}$ is in the direction of the positive z-axis. See Figure 13.31. Since we know that

$$\text{Area of the parallelogram} = \vec{c} \cdot \vec{b} = a_1b_2 - a_2b_1.$$

the definition of $\vec{a} \times \vec{b}$ tells us that

$$\vec{a} \times \vec{b} = (\text{Area of Parallelogram})\vec{k} = (\vec{c} \cdot \vec{b})\vec{k} = (a_1b_2 - a_2b_1)\vec{k}.$$

Thus,

$$\vec{a} \times \vec{b} = (a_1b_2 - a_2b_1)\vec{k}.$$

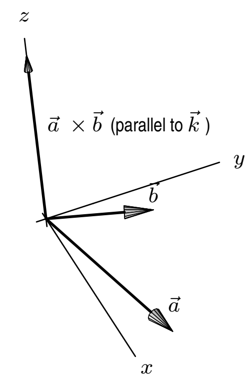

Figure 13.31: Cross product of two vectors in the xy-plane

28. If $\lambda = 0$, then all three cross products are $\vec{0}$, since the cross product of the zero vector with any other vector is always 0.

If $\lambda > 0$, then $\lambda\vec{v}$ and $\vec{v}$ are in the same direction and $\vec{w}$ and $\lambda\vec{w}$ are in the same direction. Therefore the unit normal vector $\vec{n}$ is the same in all three cases. In addition, the angles between $\lambda\vec{v}$ and $\vec{w}$, and between $\vec{v}$ and $\vec{w}$, and between $\vec{v}$ and $\lambda\vec{w}$ are all θ. Thus,

$$\begin{aligned}(\lambda\vec{v}) \times \vec{w} &= \|\lambda\vec{v}\|\|\vec{w}\| \sin\theta\vec{n} \\ &= \lambda\|\vec{v}\|\|\vec{w}\| \sin\theta\vec{n} \\ &= \lambda(\vec{v} \times \vec{w}) \\ &= \|\vec{v}\|\|\lambda\vec{w}\| \sin\theta\vec{n} \\ &= \vec{v} \times (\lambda\vec{w})\end{aligned}$$

If $\lambda < 0$, then $\lambda\vec{v}$ and $\vec{v}$ are in opposite directions, as are $\vec{w}$ and $\lambda\vec{w}$ in opposite directions. Therefore if $\vec{n}$ is the normal vector in the definition of $\vec{v} \times \vec{w}$, then the right-hand rule gives $-\vec{n}$ for $(\lambda\vec{v}) \times \vec{w}$ and $\vec{v} \times (\lambda\vec{w})$. In addition, if the angle between $\vec{v}$ and $\vec{w}$ is θ, then the angle between $\lambda\vec{v}$ and $\vec{w}$ and between $\vec{v}$ and $\lambda\vec{w}$ is $(\pi - \theta)$. Since if $\lambda < 0$, we have $|\lambda| = -\lambda$, so

$$\begin{aligned}(\lambda\vec{v}) \times \vec{w} &= \|\lambda\vec{v}\|\|\vec{w}\| \sin(\pi - \theta)(-\vec{n}) \\ &= |\lambda|\,\|\vec{v}\|\|\vec{w}\| \sin(\pi - \theta)(-\vec{n}) \\ &= -\lambda\|\vec{v}\|\|\vec{w}\| \sin\theta(-\vec{n}) \\ &= \lambda\|\vec{v}\|\|\vec{w}\| \sin\theta\vec{n} \\ &= \lambda(\vec{v} \times \vec{w}).\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{v} \times (\lambda\vec{w}) &= \|\vec{v}\|\|\lambda\vec{w}\| \sin(\pi - \theta)(-\vec{n}) \\ &= -\lambda\|\vec{v}\|\|\vec{w}\| \sin\theta(-\vec{n}) \\ &= \lambda(\vec{v} \times \vec{w}).\end{aligned}$$

29. The quantities $\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$ and $\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$ both represent the volume of the same parallelepiped, namely that defined by the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, and therefore must be equal. Thus, the two triple products $\vec{a}\cdot(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\cdot\vec{c}$ must be equal except perhaps for their sign. In fact, both are positive if $\vec{a}$, $\vec{b}$, $\vec{c}$ are right-handed and negative if $\vec{a}$, $\vec{b}$, $\vec{c}$ are left-handed. This can be shown by drawing a picture:

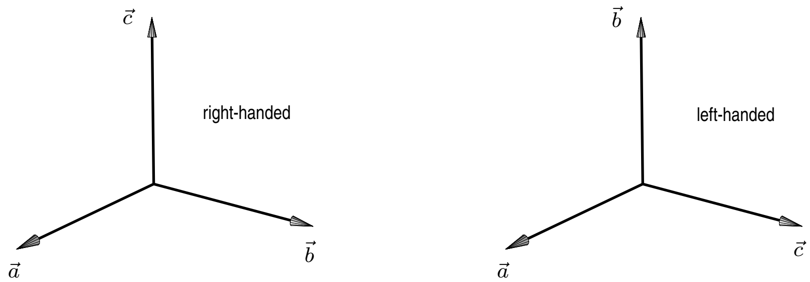

Figure 13.32

30. If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\begin{aligned}\|\vec{a}\times\vec{b}\|^2 &= (\|\vec{a}\|\|\vec{b}\|\sin\theta)^2\\ &= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta\\ &= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)\\ &= \|\vec{a}\|^2\|\vec{b}\|^2-\|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta\\ &= \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2.\end{aligned}$$

31. Write $\vec{v}$ and $\vec{w}$ in components and expand using the distributive property of the cross product.

$$\begin{aligned}\vec{v}\times\vec{w} &= (v_1\vec{i}+v_2\vec{j}+v_3\vec{k})\times(w_1\vec{i}+w_2\vec{j}+w_3\vec{k})\\ &= v_1w_1\vec{i}\times\vec{i}+v_1w_2\vec{i}\times\vec{j}+v_1w_3\vec{i}\times\vec{k}\\ &\quad+v_2w_1\vec{j}\times\vec{i}+v_2w_2\vec{j}\times\vec{j}+v_2w_3\vec{j}\times\vec{k}\\ &\quad+v_3w_1\vec{k}\times\vec{i}+v_3w_2\vec{k}\times\vec{j}+v_3w_3\vec{k}\times\vec{k}\end{aligned}$$

Now we use the fact that $\vec{i}\times\vec{i}=\vec{0}, \vec{i}\times\vec{j}=\vec{k}, \vec{i}\times\vec{k}=-\vec{j}, \vec{j}\times\vec{i}=-\vec{k}, \vec{j}\times\vec{j}=\vec{0}, \vec{j}\times\vec{k}=\vec{i}, \vec{k}\times\vec{i}=\vec{j}, \vec{k}\times\vec{j}=-\vec{i}, \vec{k}\times\vec{k}=\vec{0}$. Thus we have

$$\begin{aligned}\vec{v}\times\vec{w} &= \vec{0}+v_1w_2\vec{k}+v_1w_3(-\vec{j})+v_2w_1(-\vec{k})+\vec{0}+v_2w_3\vec{i}+v_3w_1\vec{j}+v_3w_2(-\vec{i})+\vec{0}\\ &= (v_2w_3-v_3w_2)\vec{i}+(v_3w_1-v_1w_3)\vec{j}+(v_1w_2-v_2w_1)\vec{k}.\end{aligned}$$

32. (a) Since $\vec{c}$ is perpendicular to $\vec{a}\times\vec{b}$, and since $\vec{a}\times\vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$, it follows that $\vec{c}$ must be in the plane containing $\vec{a}$ and $\vec{b}$.

(b) Using the expression given in the problem for $\vec{c}$, we get

$$\begin{aligned}\vec{a}\cdot\vec{c} &= \vec{a}\cdot(\vec{a}\times(\vec{b}\times\vec{a}))\\ &= (\vec{a}\times\vec{a})\cdot(\vec{b}\times\vec{a})\\ &= \vec{0}\cdot(\vec{b}\times\vec{a})=0.\end{aligned}$$

and

$$\begin{aligned}\vec{b}\cdot\vec{c} &= \vec{b}\cdot(\vec{a}\times(\vec{b}\times\vec{a}))\\ &= (\vec{b}\times\vec{a})\cdot(\vec{b}\times\vec{a})\\ &= \|\vec{b}\times\vec{a}\|^2\\ &= \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2.\end{aligned}$$

(c) Since $\vec{c}$ lies in the plane containing $\vec{a}$ and $\vec{b}$, it is of the form $\vec{c} = x\vec{a} + y\vec{b}$ for some scalars x and y. Thus, using the fact that $\vec{a} \cdot \vec{c} = 0$ from part (b), we have

$$\vec{a} \cdot \vec{c} = \vec{a} \cdot (x\vec{a} + y\vec{b}) = x\|\vec{a}\|^2 + y(\vec{a} \cdot \vec{b}) = 0.$$

Similarly, using the fact that $\vec{b} \cdot \vec{c} = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$ from part (b), we have

$$\vec{b} \cdot \vec{c} = \vec{b} \cdot (x\vec{a} + y\vec{b}) = x(\vec{a} \cdot \vec{b}) + y\|\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2.$$

Solving these two linear equations in x and y, we find $x = -\vec{a} \cdot \vec{b}$ and $y = \|\vec{a}\|^2$.

33. Problem 29 tells us that $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$. Using this result on the triple product of $(\vec{a} + \vec{b}) \times \vec{c}$ with any vector $\vec{d}$ together with the fact that the dot product distributes over addition gives us:

$$\begin{aligned}[(\vec{a} + \vec{b}) \times \vec{c}] \cdot \vec{d} &= (\vec{a} + \vec{b}) \cdot (\vec{c} \times \vec{d}) \\ &= \vec{a} \cdot (\vec{c} \times \vec{d}) + \vec{b} \cdot (\vec{c} \times \vec{d}) \qquad \text{(dot product is distributive)} \\ &= (\vec{a} \times \vec{c}) \cdot \vec{d} + (\vec{b} \times \vec{c}) \cdot \vec{d} \qquad \text{(using Problem 29 again)} \\ &= [(\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})] \cdot \vec{d}. \qquad \text{(dot product is distributive)}\end{aligned}$$

So, since $[(\vec{a} + \vec{b}) \times \vec{c}] \cdot \vec{d} = [(\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})] \cdot \vec{d}$, then

$$[(\vec{a} + \vec{b}) \times \vec{c}] \cdot \vec{d} - [(\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})] \cdot \vec{d} = 0,$$

Since the dot product is distributive, we have

$$[((\vec{a} + \vec{b}) \times \vec{c}) - (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{c})] \cdot \vec{d} = 0.$$

Since this equation is true for all vectors $\vec{d}$, by letting

$$\vec{d} = ((\vec{a} + \vec{b}) \times \vec{c}) - (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{c}),$$

we get

$$\|(\vec{a} + \vec{b}) \times \vec{c} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c}\|^2 = 0$$

and hence

$$(\vec{a} + \vec{b}) \times \vec{c} - (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{c}) = \vec{0}.$$

Thus

$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}).$$

34. The area vector for face $OAB = \frac{1}{2}\vec{b} \times \vec{a}$.
The area vector for face $OBC = \frac{1}{2}\vec{a} \times \vec{c}$.
The area vector for face $OAC = \frac{1}{2}\vec{b} \times \vec{c}$.
The area vector for face $ABC = \frac{1}{2}(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$.

$$\frac{1}{2}\vec{b} \times \vec{a} + \frac{1}{2}\vec{c} \times \vec{b} + \frac{1}{2}\vec{a} \times \vec{c} + \frac{1}{2}(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) =$$
$$\frac{1}{2}\vec{b} \times \vec{a} + \frac{1}{2}\vec{c} \times \vec{b} + \frac{1}{2}\vec{a} \times \vec{c} + \frac{1}{2}(\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} - \vec{a} \times \vec{a}) = 0.$$

Solutions for Chapter 13 Review

Exercises

1. $\vec{v} + 2\vec{w} = 2\vec{i} + 3\vec{j} - \vec{k} + 2(\vec{i} - \vec{j} + 2\vec{k}) = 4\vec{i} + \vec{j} + 3\vec{k}$.

2. $3\vec{v} - \vec{w} - \vec{v} = 2\vec{v} - \vec{w} = 2(2\vec{i} + 3\vec{j} - \vec{k}) - (\vec{i} - \vec{j} + 2\vec{k}) = 3\vec{i} + 7\vec{j} - 4\vec{k}$.

3. $\vec{v}\cdot\vec{w} = (2\vec{i}+3\vec{j}-\vec{k})\cdot(\vec{i}-\vec{j}+2\vec{k}) = 2\cdot 1 + 3\cdot(-1) + (-1)\cdot 2 = -3.$

4. $\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix} = (6-1)\vec{i} - (4+1)\vec{j} + (-2-3)\vec{k} = 5\vec{i} - 5\vec{j} - 5\vec{k}.$

5. For any vector $\vec{v}$, we have $\vec{v}\times\vec{v} = \vec{0}$.

6. $||\vec{v}+\vec{w}|| = ||3\vec{i}+2\vec{j}+\vec{k}|| = \sqrt{3^2+2^2+1^2} = \sqrt{14}.$

7. Since $\vec{v}\cdot\vec{w} = 2\cdot 1 + 3(-1) + (-1)2 = -3$, we have $(\vec{v}\cdot\vec{w})\vec{v} = -6\vec{i} - 9\vec{j} + 3\vec{k}$.

8. We have $\vec{v}\times\vec{w} = 5\vec{i} - 5\vec{j} - 5\vec{k}$, so

$$(\vec{v}\times\vec{w})\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -5 & -5 \\ 1 & -1 & 2 \end{vmatrix} = (-10-5)\vec{i} - (10+5)\vec{j} + (-5+5)\vec{k} = -15\vec{i} - 15\vec{j}.$$

9. Since $\vec{v}\times\vec{w}$ is perpendicular to $\vec{w}$, we have $(\vec{v}\times\vec{w})\cdot\vec{w} = 0$.

10. The cross product of two parallel vectors is $\vec{0}$, so the cross product of any vector with itself is $\vec{0}$.

11. (a) We have $\vec{v}\cdot\vec{w} = 3\cdot 4 + 2\cdot(-3) + (-2)\cdot 1 = 4$.

(b) We have $\vec{v}\times\vec{w} = -4\vec{i} - 11\vec{j} - 17\vec{k}$.

(c) A vector of length 5 parallel to $\vec{v}$ is

$$\frac{5}{||\vec{v}||}\vec{v} = \frac{5}{\sqrt{17}}(3\vec{i}+2\vec{j}-2\vec{k}) = 3.64\vec{i} + 2.43\vec{j} - 2.43\vec{k}.$$

(d) The angle between vectors $\vec{v}$ and $\vec{w}$ is found using

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||||\vec{w}||} = \frac{4}{\sqrt{17}\sqrt{26}} = 0.190,$$

so $\theta = 79.0^\circ$.

(e) The component of vector $\vec{v}$ in the direction of vector $\vec{w}$ is

$$\frac{\vec{v}\cdot\vec{w}}{||\vec{w}||} = \frac{4}{\sqrt{26}} = 0.784.$$

(f) The answer is any vector $\vec{a}$ such that $\vec{a}\cdot\vec{v} = 0$. One possible answer is $2\vec{i} - 2\vec{j} + \vec{k}$.

(g) A vector perpendicular to both is the cross product:

$$\vec{v}\times\vec{w} = -4\vec{i} - 11\vec{j} - 17\vec{k}.$$

12. Since $||2\vec{i}+3\vec{j}-\vec{k}|| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14}$, vectors of length 10 are

$$\pm\frac{10}{\sqrt{14}}(2\vec{i}+3\vec{j}-\vec{k}).$$

13. We take the cross product of $\vec{i}+\vec{j}$ and $\vec{i}-\vec{j}-\vec{k}$ and then make a unit vector parallel to the cross product.

$$(\vec{i}+\vec{j})\times(\vec{i}-\vec{j}-\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -\vec{i} + \vec{j} - 2\vec{k}.$$

Since $||-\vec{i}+\vec{j}-2\vec{k}|| = \sqrt{(-1)^2+1^2+(-2)^2} = 6$, unit vectors are

$$\pm\frac{-\vec{i}+\vec{j}-2\vec{k}}{\sqrt{6}}.$$

14. We want a unit vector of the form $a\vec{i} + b\vec{j}$ such that

$$(a\vec{i} + b\vec{j}) \cdot (3\vec{i} - 2\vec{j}) = 3a - 2b = 0.$$

Let's take $a = 2$ and $b = 3$. Then the vector $2\vec{i} + 3\vec{j}$ is perpendicular to $3\vec{i} - 2\vec{j}$, but $2\vec{i} + 3\vec{j}$ is not a unit vector. Since $\|2\vec{i} + 3\vec{j}\| = \sqrt{13}$, unit vectors are

$$\pm\frac{2\vec{i} + 3\vec{j}}{\sqrt{13}}.$$

15. The vector $\vec{w}$ we want is shown in Figure 13.33, where the given vector is $\vec{v} = 4\vec{i} + 3\vec{j}$. The vectors $\vec{v}$ and $\vec{w}$ are the same length and the two angles marked α are equal, so the two right triangles shown are congruent. Thus

$$a = -3 \qquad \text{and} \qquad b = 4.$$

Therefore

$$\vec{w} = -3\vec{i} + 4\vec{j}.$$

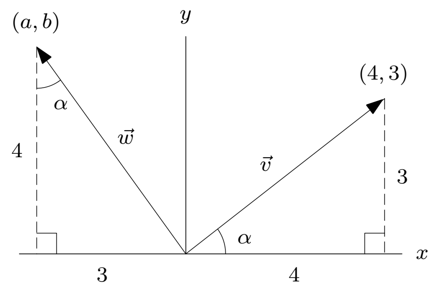

Figure 13.33

Problems

16. (a) We need $6\vec{i} + 8\vec{j} + 3\vec{k} = \lambda(2\vec{i} + (t^2 + \frac{2}{3}t + 1)\vec{j} + t\vec{k})$ for some λ. This gives

$$\begin{aligned} 6 &= 2\lambda \\ 8 &= (t^2 + \frac{2}{3}t + 1)\lambda \\ 3 &= t\lambda \end{aligned}$$

From the first equation, we have $\lambda = 3$. Substituting $\lambda = 3$ into the third equation gives $t = 1$. Check the second equation, it says $8 = 8$, if $t = 1$ and $\lambda = 3$. So for $t = 1$, the two vectors are parallel to each other.

(b) Similar to part (a), we need to solve

$$\begin{aligned} 2 &= t\lambda \\ -4 &= \lambda \\ 1 &= \lambda(t - 1) \end{aligned}$$

From the first two equations we have $\lambda = -4$ and $t = -\frac{1}{2}$. Substituting this into the third equation gives $1 = 6$. Thus this system of equations has no solution, so the pair of vectors is not parallel to each other for any value of t.

(c) $2t\vec{i} + t\vec{j} + t\vec{k} = \frac{t}{3}(6\vec{i} + 3\vec{j} + 3\vec{k})$. For any t, the two vectors are parallel to each other.

17. (a) Since $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ and $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$, we find

$$\|\vec{v} \times \vec{w}\| = \|12\vec{i} - 3\vec{j} + 4\vec{k}\| = \sqrt{12^2 + (-3)^2 + 4^2} = 13.$$

Then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\|\vec{v} \times \vec{w}\|}{\vec{v} \cdot \vec{w}} = \frac{13}{8} = 1.625.$$

(b) Then $\theta = \tan^{-1}(1.625) = 1.019$.

18. $\vec{n} = 4\vec{i} + 6\vec{k}$ (the coefficients of x, y, z are the same as the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$.)

19. If the planes are parallel, they have a common normal vector $\vec{n}$. Rewrite the equation of the plane as $4x - 3y - z = -8$ so that $\vec{n} = 4\vec{i} - 3\vec{j} - \vec{k}$ and the desired plane is $4(x-0) - 3(y-0) - (z-0) = 0$ or $4x - 3y - z = 0$.

20. (a) On the x-axis, $y = z = 0$, so $5x = 21$, giving $x = \frac{21}{5}$. So the only such point is $(\frac{21}{5}, 0, 0)$.
(b) Other points are $(0, -21, 0)$, and $(0, 0, 3)$. There are many other possible answers.
(c) $\vec{n} = 5\vec{i} - \vec{j} + 7\vec{k}$. It is the normal vector.
(d) The vector between two points in the plane is parallel to the plane. Using the points from part (b), the vector $3\vec{k} - (-21\vec{j}) = 21\vec{j} + 3\vec{k}$ is parallel to the plane.

21. Let $\vec{r}_1$ be the displacement vector $\overrightarrow{PQ}$ and let $\vec{r}_2$ be the displacement vector $\overrightarrow{PR}$. Then

$$\vec{r}_1 = (1+2)\vec{i} + (3-2)\vec{j} + (-1-0)\vec{k} = 3\vec{i} + \vec{j} - \vec{k},$$
$$\vec{r}_2 = (-4+2)\vec{i} + (2-2)\vec{j} + (1-0)\vec{k} = -2\vec{i} + \vec{k},$$
$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = \vec{i} - (3-2)\vec{j} + 2\vec{k} = \vec{i} - \vec{j} + 2\vec{k}.$$

The area of the triangle $= \frac{1}{2}\|\vec{r}_1 \times \vec{r}_2\| = \frac{1}{2}\sqrt{1^2 + 1^2 + 2^2} = \frac{\sqrt{6}}{2}$.

22. (a) The displacement vector $\overrightarrow{AB}$ lies in the plane and is given by

$$\overrightarrow{AB} = (0-2)\vec{i} + (1-1)\vec{j} + (3-0)\vec{k} = -2\vec{i} + 3\vec{k}.$$

Similarly, the displacement vector $\overrightarrow{AC}$ also lies in the plane,

$$\overrightarrow{AC} = (1-2)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = -\vec{i} - \vec{j} + \vec{k}.$$

(b) The vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and is therefore perpendicular to the plane.

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-2\vec{i} + 3\vec{k}) \times (-\vec{i} - \vec{j} + \vec{k}) = 3\vec{i} - \vec{j} + 2\vec{k}.$$

(c) The normal vector to the plane is $\vec{n} = 3\vec{i} - \vec{j} + 2\vec{k}$, so the equation is of the form

$$3x - y + 2z = d.$$

Substituting, for example, $x = 1$, $y = 0$, $z = 1$ gives $d = 5$:

$$3x - y + 2z = 5.$$

23. (a) Since

$$\overrightarrow{PQ} = (3\vec{i} + 5\vec{j} + 7\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 2\vec{i} + 3\vec{j} + 4\vec{k},$$

and

$$\overrightarrow{PR} = (2\vec{i} + 5\vec{j} + 3\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = \vec{i} + 3\vec{j},$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 1 & 3 & 0 \end{vmatrix} = -12\vec{i} + 4\vec{j} + 3\vec{k},$$

which is a vector perpendicular to the plane containing P, Q and R. Since

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(-12)^2 + 4^2 + 3^2} = 13,$$

the unit vectors which are perpendicular to a plane containing P, Q, and R are

$$-\frac{12}{13}\vec{i} + \frac{4}{13}\vec{j} + \frac{3}{13}\vec{k},$$

or the unit vector pointing to the opposite direction,

$$\frac{12}{13}\vec{i} - \frac{4}{13}\vec{j} - \frac{3}{13}\vec{k}.$$

(b) The angle between PQ and PR is θ for which

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\cdot\|\overrightarrow{PR}\|} = \frac{2\cdot 1+3\cdot 3+4\cdot 0}{\sqrt{2^2+3^2+4^2}\cdot\sqrt{1^2+3^2+0^2}} = \frac{11}{\sqrt{290}},$$

so

$$\theta = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.76^\circ.$$

(c) The area of triangle $PQR = \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \frac{13}{2}$.

(d) Let d be the distance from R to the line through P and Q (see Figure 13.34), then

$$\frac{1}{2}d\cdot\|\overrightarrow{PQ}\| = \text{the area of } \triangle\, PQR = \frac{13}{2}.$$

Therefore,

$$d = \frac{13}{\|\overrightarrow{PQ}\|} = \frac{13}{\sqrt{2^2+3^2+4^2}} = \frac{13}{\sqrt{29}}.$$

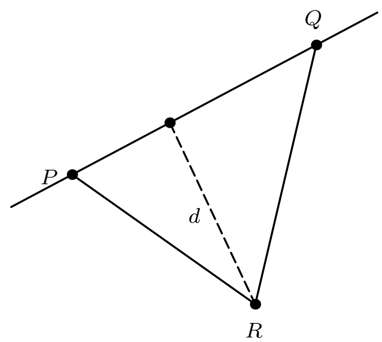

Figure 13.34

24. Find an arbitrary point on the plane $2x+4y-z=-1$, say $A=(0,0,1)$. The normal $\vec{n}$ to the plane at B is $\vec{n} = 2\vec{i}+4\vec{j}-\vec{k}$ and $\overrightarrow{PA} = -2\vec{i}+\vec{j}-2\vec{k}$. See Figure 13.35.

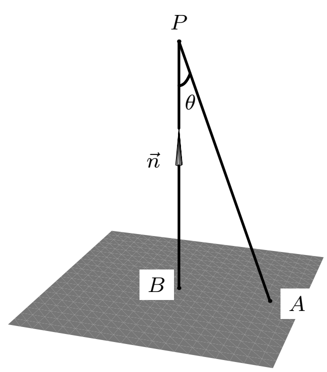

Figure 13.35

So the distance d from the point P to the plane is

$$\begin{aligned} d &= \|\overrightarrow{PB}\| = \|\overrightarrow{PA}\|\cos\theta \\ &= \frac{\overrightarrow{PA}\cdot\vec{n}}{\|\vec{n}\|} \quad \text{since } \overrightarrow{PA}\cdot\vec{n} = \|\overrightarrow{PA}\|\|\vec{n}\|\cos\theta) \\ &= \frac{(-2\vec{i}+\vec{j}-2\vec{k})\cdot(2\vec{i}+4\vec{j}-\vec{k})}{\sqrt{2^2+4^2+(-1)^2}} \\ &= \frac{2}{\sqrt{21}}. \end{aligned}$$

25. The displacement from $(1,1,1)$ to $(1,4,5)$ is

$$\vec{r_1} = (1-1)\vec{i} + (4-1)\vec{j} + (5-1)\vec{k} = 3\vec{j} + 4\vec{k}.$$

The displacement from $(-3,-2,0)$ to $(1,4,5)$ is

$$\vec{r_2} = (1+3)\vec{i} + (4+2)\vec{j} + (5-0)\vec{k} = 4\vec{i} + 6\vec{j} + 5\vec{k}.$$

A normal vector is

$$\vec{n} = \vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 4 \\ 4 & 6 & 5 \end{vmatrix} = (15-24)\vec{i} - (-16)\vec{j} + (-12)\vec{k} = -9\vec{i} + 16\vec{j} - 12\vec{k}.$$

The equation of the plane is

$$\begin{aligned} -9x + 16y - 12z &= -9 \cdot 1 + 16 \cdot 1 - 12 \cdot 1 = -5 \\ 9x - 16y + 12z &= 5. \end{aligned}$$

We pick a point A on the plane, $A = (\frac{5}{9}, 0, 0)$ and let $P = (0,0,0)$. (See Figure 13.36.) Then $\vec{PA} = (5/9)\vec{i}$.

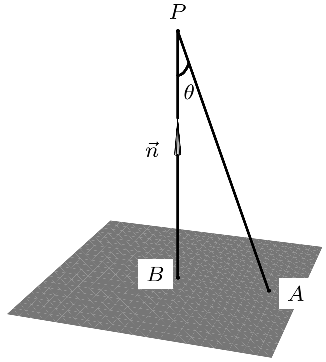

Figure 13.36

So the distance d from the point P to the plane is

$$\begin{aligned} d &= \|\overrightarrow{PB}\| = \|\overrightarrow{PA}\| \cos\theta \\ &= \frac{\overrightarrow{PA} \cdot \vec{n}}{\|\vec{n}\|} \quad \text{since } \overrightarrow{PA} \cdot \vec{n} = \|\overrightarrow{PA}\| \|\vec{n}\| \cos\theta) \\ &= \left| \frac{(\frac{5}{9}\vec{i}) \cdot (-9\vec{i} + 16\vec{j} - 12\vec{k})}{\sqrt{9^2 + 16^2 + 12^2}} \right| \\ &= \frac{5}{\sqrt{481}} = 0.23. \end{aligned}$$

26. Suppose $\vec{u}$ represents the velocity of the plane relative to the air and $\vec{w}$ represents the velocity of the wind. We can add these two vectors by adding their components. Suppose north is in the y-direction and east is the x-direction. The vector representing the airplane's velocity makes an angle of 45° with north; the components of $\vec{u}$ are

$$\vec{u} = 700 \sin 45^\circ \vec{i} + 700 \cos 45^\circ \vec{j} \approx 495\vec{i} + 495\vec{j}.$$

Since the wind is blowing from the west, $\vec{w} = 60\vec{i}$. By adding these we get a resultant vector $\vec{v} = 555\vec{i} + 495\vec{j}$. The direction relative to the north is the angle θ shown in Figure 13.37 given by

$$\begin{aligned} \theta &= \tan^{-1} \frac{x}{y} = \tan^{-1} \frac{555}{495} \\ &\approx 48.3^\circ \end{aligned}$$

The magnitude of the velocity is

$$\begin{aligned}\|\vec{v}\| &= \sqrt{495^2 + 555^2} = \sqrt{553{,}050} \\ &= 744 \text{ km/hr}.\end{aligned}$$

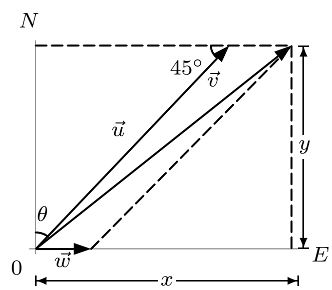

Figure 13.37: Note that θ is the angle between north and the vector $\vec{v}$

27. (a) Let x-axis be the East direction and y-axis be the North direction. From Figure 13.38,

$$\theta = \sin^{-1}(4/5) = 53.1^\circ.$$

That is, he should steer at 53.1° east of south.

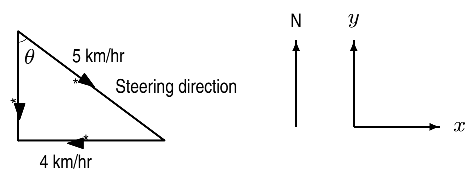

Figure 13.38

(b)

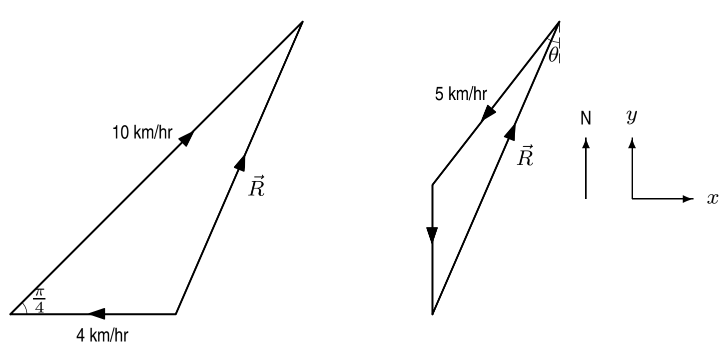

Figure 13.39

Let $\vec{R}$ be the resultant of the wind and river velocities, that is

$$\begin{aligned}\vec{R} &= -4\vec{i} + (10\cos(\frac{\pi}{4})\vec{i} + 10\cos(\frac{\pi}{4})\vec{j}) \\ &= (-4 + 5\sqrt{2})\vec{i} + 5\sqrt{2}\vec{j}.\end{aligned}$$

From Figure 13.39, we see that to get the the x-component of his rowing velocity and the x-component of $\vec{R}$ to cancel each other, we must have

$$5\sin\theta = -4 + 5\sqrt{2}$$
$$\theta = \sin^{-1}\left(\frac{-4+5\sqrt{2}}{5}\right) = 37.9^\circ.$$

However for this value of θ, the y-component of the velocity is

$$5\sqrt{2} - 5\cos(37.9^\circ) = 3.1.$$

Since the y-component is positive, the man will not move across the river in a southward direction.

28. The speed of the particle before impact is v, so the speed after impact is $0.8v$. If we consider the barrier as being along the x-axis (see Figure 13.40), then the $\vec{i}$-component is $0.8v\cos 60^\circ = 0.8v(0.5) = 0.4v$.
Similarly, the $\vec{j}$-component is $0.8v\sin 60^\circ = 0.8v(0.8660) \approx 0.7v$. Thus

$$\vec{v}_{\text{after}} = 0.4v\vec{i} + 0.7v\vec{j}.$$

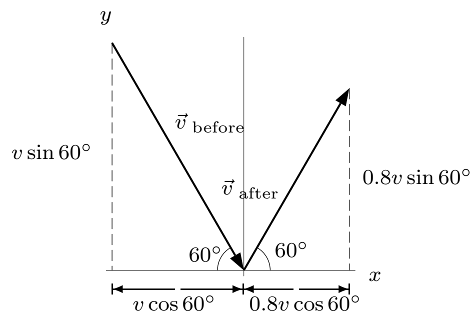

Figure 13.40

29. (a) 500 km/hr in the west direction, so $\vec{v} = -500\vec{i}$.
(b) While traveling at constant altitude, the plane travels 250 km westward. Thus the coordinates of the point where the plane begins to descend are $(550, 60, 4) - (250, 0, 0) = (300, 60, 4)$.
(c) The vector from the plane to the airport at the time it begins its descent is $(200\vec{i} + 10\vec{j}) - (300\vec{i} + 60\vec{j} + 4\vec{k}) = -100\vec{i} - 50\vec{j} - 4\vec{k}$. Velocity is a vector of length 200 km/hr in the direction of $-100\vec{i} - 50\vec{j} - 4\vec{k}$. Since $\sqrt{(-100)^2 + (-50)^2 + (-4)^2} \approx 111.9$, a unit vector in the direction of descent is $-\frac{100}{111.9}\vec{i} - \frac{50}{111.9}\vec{j} - \frac{4}{111.9}\vec{k}$. Thus

$$\text{Velocity vector} = 200\left(-\frac{100}{111.9}\vec{i} - \frac{50}{111.9}\vec{j} - \frac{4}{111.9}\vec{k}\right) = -178.7\vec{i} - 89.4\vec{j} - 7.2\vec{k}.$$

30. (a) The displacement vector of the moon relative to the earth is

$$\vec{r} = 384\vec{i}.$$

The displacement vector of the spaceship relative to the earth is

$$\vec{r_E} = 280\vec{i} + 90\vec{j}.$$

The displacement vector of the spaceship relative to the moon is

$$\vec{r_L} = \vec{r_E} - \vec{r} = -104\vec{i} + 90\vec{j}.$$

See Figure 13.41.
(b) Distance of spaceship from Earth $= ||\vec{r_E}|| = \sqrt{280^2 + 90^2} = \sqrt{86500} = 294.109$ thousand km.
Distance of spaceship from the moon $= ||\vec{r_L}|| = \sqrt{(-104)^2 + 90^2} = \sqrt{18916} = 137.535$ thousand km.

(c) See Figure 13.41. The gravitational force of the earth, $\overrightarrow{F_E}$, is parallel to $\vec{r_E}$ but of length 461 and in the opposite direction:

$$\overrightarrow{F_E} = -\frac{461}{\sqrt{186500}}(280\vec{i} + 90\vec{j}) = -438.885\vec{i} - 141.070\vec{j}.$$

The gravitational force of the moon, $\overrightarrow{F_L}$, is parallel to $\vec{r_L}$ but of length 26 and in the opposite direction:

$$\overrightarrow{F_L} = -\frac{26}{\sqrt{18916}}(-104\vec{i} + 90\vec{j}) = 19.660\vec{i} - 17.041\vec{j}.$$

The resulting force, $\vec{F}$ is

$$\vec{F} = \overrightarrow{F_E} + \overrightarrow{F_L} = 419.225\vec{i} - 158.084\vec{j}.$$

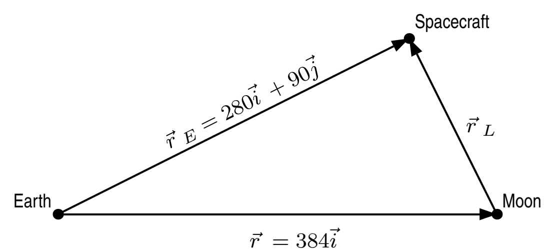

Figure 13.41

31. Let $\vec{R}$ be the resultant force, and let $\vec{F}_1$ and $\vec{F}_2$ be the forces exerted by the larger and smaller tugs. See Figure 13.42. Then $\|\vec{F_1}\| = \frac{5}{4}\|\vec{F_2}\|$. The y components of the vectors $\vec{F}_1$ and $\vec{F}_2$ must cancel each other in order to ensure that the ship travels due east, hence

$$\|\vec{F_1}\| \sin 30^\circ = \|\vec{F_2}\| \sin\theta,$$

so

$$\frac{5}{4}\|\vec{F_2}\| \sin 30^\circ = \|\vec{F_2}\| \sin\theta,$$

giving $\sin\theta = \frac{5}{8}$, and hence $\theta = \sin^{-1}\frac{5}{8} = 38.7^\circ$.

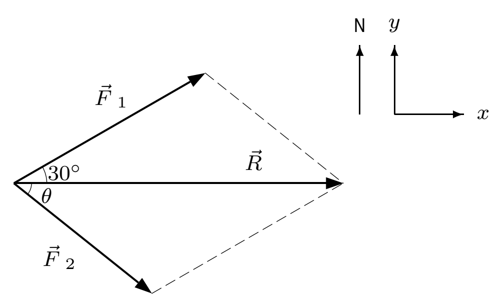

Figure 13.42

32. Let the x-axis point east and the y-axis point north. We use $\vec{C}$, $\vec{W}$, and $\vec{E}$ to represent the current, wind, and engine vectors, respectively. We resolve the current and wind velocity vectors into components. Since the current points 25° north of east with a speed of 12, we have

$$\vec{C} = 12\cos(25^\circ)\vec{i} + 12\sin(25^\circ)\vec{j} = 10.876\vec{i} + 5.071\vec{j}.$$

Since $\vec{C}$ lies in the first quadrant, both coefficients are positive.

The wind points 80° south of east with a speed of 7 km/hr, so we have

$$\vec{W} = 7\cos(80^\circ)\vec{i} - 7\sin(80^\circ)\vec{j} = 1.216\vec{i} - 6.894\vec{j}.$$

Since $\vec{W}$ lies in the fourth quadrant, the coefficient of $\vec{i}$ is positive and the coefficient of $\vec{j}$ is negative.

The combined velocity on the boat is due east at a speed of 40 km/hr, so we want

$$\vec{C} + \vec{W} + \vec{E} = 40\vec{i}.$$

We solve for $\vec{E}$:

$$\begin{aligned}\vec{E} &= 40\vec{i} - (\vec{C} + \vec{W})\\ &= 40\vec{i} - ((10.876\vec{i} + 5.071\vec{j}) + (1.216\vec{i} - 6.894\vec{j}))\\ &= 40\vec{i} - (12.092\vec{i} - 1.823\vec{j})\\ &= 27.908\vec{i} + 1.823\vec{j}.\end{aligned}$$

The engine should push the boat with a speed of $\|\vec{E}\| = \sqrt{27.908^2 + 1.823^2} = 27.97$ km/hr, and in direction $\arctan(1.823/27.908) = 3.74°$ north of east.

33. Let the x-axis point east and the y-axis point north. Denote the forces exerted by Charlie, Sam and Alice by $\vec{F}_C, \vec{F}_S$ and $\vec{F}_A$ (see Figure 13.43).

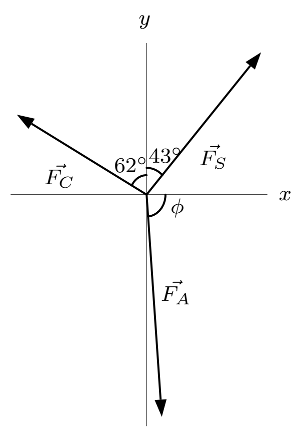

Figure 13.43

Since $\|\vec{F}_C\| = 175$ newtons and the angle θ from the x-axis to $\vec{F}_C$ is $90° + 62° = 152°$, we have

$$\vec{F}_C = 175\cos 152°\vec{i} + 175\sin 152°\vec{j} \approx -154.52\vec{i} + 82.16\vec{j}.$$

Similarly,

$$\vec{F}_S = 200\cos 47°\vec{i} + 200\sin 47°\vec{j} \approx 136.4\vec{i} + 146.27\vec{j}.$$

Now Alice is to counterbalance Sam and Charlie, so the resultant force of the three forces $\vec{F}_C, \vec{F}_S$ and $\vec{F}_A$ must be 0, that is,

$$\vec{F}_C + \vec{F}_S + \vec{F}_A = 0.$$

Thus, we have

$$\begin{aligned}\vec{F}_A &= -\vec{F}_C - \vec{F}_S\\ &\approx -(-154.52\vec{i} + 82.16\vec{j}) - (136.4\vec{i} + 146.27\vec{j})\\ &= 18.12\vec{i} - 228.43\vec{j}\end{aligned}$$

and, $\|\vec{F}_A\| = \sqrt{18.12^2 + (-228.43)^2} \approx 229.15$ newtons.

If ϕ is the angle from the x-axis to $\vec{F}_A$, then

$$\phi = \arctan\frac{-228.43}{18.12} \approx -85.5°.$$

34. (a) Since the radius of the circle is 1 meter, the circumference is 2π meters. Thus, the object is moving at 2π meters/minute, or $\pi/30$ meters/second ≈ 0.11 meters/second.

(b) 30 seconds after passing the point $(0,1)$, the object is at the point $(-1,0)$. (Since it completes 1 revolution each minute, it will move π radians in 30 seconds.) This is true regardless of whether the point is moving clockwise or counterclockwise. However, since the velocity vector, $\vec{v}$, is tangential to the curve in the direction of motion, it will have an opposite sign if the motion is in the opposite direction. So, moving clockwise $\vec{v} = 2\pi\vec{j}$, and moving counterclockwise $\vec{v} = -2\pi\vec{j}$, if the speed is measured in meters/minute.

35. The speed is a scalar which equals 30 times the circumference of the circle per minute. So it is a constant. The velocity is a vector. Since the direction of the motion changes all the time, the velocity is not constant. This implies that the acceleration is nonzero.

36. Let $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ be the vector. We will use the properties given in the problem to find v_x, v_y, and v_z. If $\vec{v}$ has magnitude 10, then $\|\vec{v}\| = 10$.

If $\vec{v}$ makes an angle of 45° with the x-axis, then its x-component, v_x, is given by:

$$v_x = \vec{v}\cdot\vec{i} = \|\vec{v}\|\cos 45^\circ = 10\left(\frac{\sqrt{2}}{2}\right) = 7.0710.$$

Similarly, if $\vec{v}$ makes a 75° angle with the y-axis, then its y-component, v_y, is given by:

$$v_y = \vec{v}\cdot\vec{j} = \|\vec{v}\|\cos 75^\circ = 10(0.25882) = 2.5882.$$

We now have two components of $\vec{v}$:

$$\vec{v} = 7.0710\vec{i} + 2.5882\vec{j} + v_z\vec{k}.$$

We only need to find v_z. To do this we use the fact that $\sqrt{\vec{v}\cdot\vec{v}} = \|\vec{v}\| = 10$.

$$\begin{aligned}\vec{v}\cdot\vec{v} &= 100\\ v_x^2 + v_y^2 + v_z^2 &= 100\\ v_z^2 &= 100 - v_x^2 - v_y^2\\ v_z^2 &= \pm\sqrt{100 - v_x^2 - v_y^2}\\ v_z &= \pm 6.580\end{aligned}$$

Since the problem tells us that the $\vec{k}$-component is positive, $v_z = +6.580$. Thus

$$\vec{v} = 7.0710\vec{i} + 2.5882\vec{j} + 6.580\vec{k}.$$

37. (a)

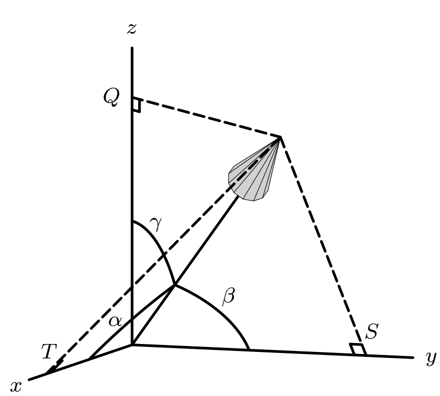

Figure 13.44

Suppose $\vec{v} = \overrightarrow{OP}$ as in Figure 13.44. The $\vec{i}$ component of $\overrightarrow{OP}$ is the projection of $\overrightarrow{OP}$ on the x-axis:

$$\overrightarrow{OT} = v\cos\alpha\vec{i}.$$

Similarly, the $\vec{j}$ and $\vec{k}$ components of $\overrightarrow{OP}$ are the projections of $\overrightarrow{OP}$ on the y-axis and the z-axis respectively. So:

$$\overrightarrow{OS} = v\cos\beta\vec{j}$$
$$\overrightarrow{OQ} = v\cos\gamma\vec{k}$$

Since $\vec{v} = \overrightarrow{OT} + \overrightarrow{OS} + \overrightarrow{OQ}$, we have

$$\vec{v} = v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}\,.$$

(b) Since

$$\begin{aligned} v^2 = \vec{v}\cdot\vec{v} &= (v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k})\cdot \\ &\quad (v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}) \\ &= v^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \end{aligned}$$

so

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

38. In Figure 13.45, let l_1 be a line with direction vector $\vec{v_1}$ passing through P_1. Let l_2 be a line with direction vector $\vec{v_2}$ passing through P_2. Lines l_1 and l_2 are skew if they are not parallel and do not intersect.

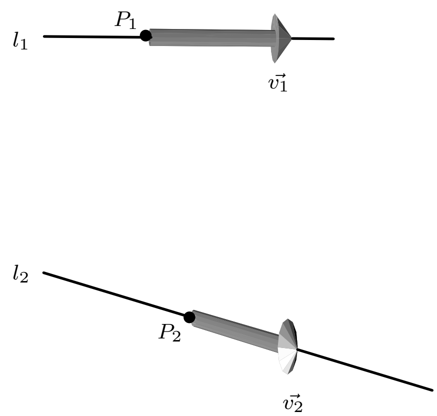

Figure 13.45

If we draw a line l_3 parallel to l_1, i.e., with a direction vector v_1, passing through P_2, as shown in Figure 13.46, then l_2 and l_3 determine a plane that is parallel to l_1:

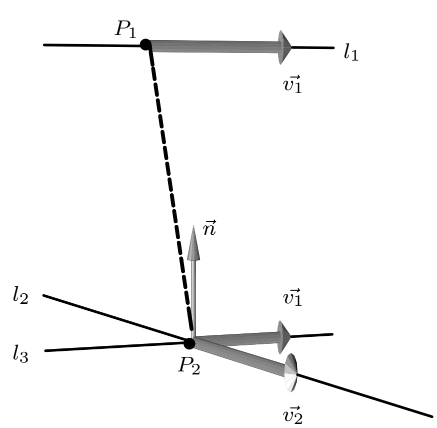

Figure 13.46

The minimum distance between l_1 and l_2 is equal to the distance of l_1 to the plane. So it is equal to the projection of $\overrightarrow{P_1P_2}$ in the direction of the normal vector of the plane. The unit normal vector is given by:

$$\vec{n} = \frac{\vec{v}_1 \times \vec{v}_2}{||\vec{v}_1 \times \vec{v}_2||},$$

so the component of $\overrightarrow{P_1P_2}$ in the direction of $\vec{n}$ is:

$$\overrightarrow{P_1P_2} \cdot \vec{n} = \overrightarrow{P_1P_2} \cdot \frac{\vec{v}_1 \times \vec{v}_2}{||\vec{v}_1 \times \vec{v}_2||}.$$

Thus the minimum distance between l_1 and l_2 is:

$$\frac{|\overrightarrow{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2)|}{||\vec{v}_1 \times \vec{v}_2||}.$$

CAS Challenge Problems

39. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = \vec{0}$

Since $\vec{c}$ is the sum of a scalar multiple of $\vec{a}$ and a scalar multiple of $\vec{b}$, it lies in the plane containing $\vec{a}$ and $\vec{b}$. On the other hand, $\vec{a} \times \vec{b}$ is perpendicular to this plane, so $\vec{a} \times \vec{b}$ is perpendicular to $\vec{c}$. Therefore, $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. Also, $\vec{a} \times \vec{c}$ is also perpendicular to the plane, thus parallel to $\vec{a} \times \vec{b}$, and thus $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = \vec{0}$.

40. The first parallelepiped has volume

$$|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |ywr - vzr + zus - xws + xvt - yut|.$$

The second has volume $|(\vec{a} \times \vec{b}) \cdot (2\vec{a} - \vec{b} + \vec{c})|$, which also simplifies to $|ywr - vzr + zus - xws + xvt - yut|$. Both parallelepipeds have base with edges $\vec{a}$ and $\vec{b}$. The third edge of the first one is $\vec{c}$ and the third edge of the second one is $\vec{c} + 2\vec{a} - \vec{b}$. Thus the top face of the second parallelepiped is obtained by shifting the top face of the first by $2\vec{a} - \vec{b}$. Since this is parallel to the base, the second parallelepiped has the same altitude as the first. Since the volume of a parallelepiped is product of the area of its base with its height, the two parallelepipeds have the same volume.

41. (a) From the geometric definition of the dot product, we have

$$\cos\theta = \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\|\|\vec{b}\|} = \frac{10}{\sqrt{14}\sqrt{9}}.$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we get

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + y + 2z &= 0 \\ x^2 + y^2 + z^2 &= \|\vec{a}\|^2\|\vec{b}\|^2(1 - \cos^2\theta) = (14)(9)\left(1 - \frac{100}{(14)(9)}\right) \end{aligned}$$

Solving these equations we get $x = -1, y = -4, z = 3$ or $x = 1, y = 4$, and $z = -3$. Thus $\vec{c} = -\vec{i} - 4\vec{j} + 3\vec{k}$ or $\vec{c} = \vec{i} + 4\vec{j} - 3\vec{k}$.

(b) $\vec{a} \times \vec{b} = \vec{i} + 4\vec{j} - 3\vec{k}$. This is the same as one of the answers in part (a). The conditions in part (a) ensured that $\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$ and that it has magnitude $\|\vec{a}\|\|\vec{b}\||\sin\theta|$. The cross product is the solution that, in addition, satisfies the right-hand rule.

42. (a) We have

$$\begin{aligned} \|\overrightarrow{AB}\| &= \|2\vec{i}\| = 2 \\ \|\overrightarrow{AC}\| &= \|\vec{i} + \sqrt{3}\vec{j}\| = \sqrt{1+3} = 2 \\ \|\overrightarrow{AD}\| &= \|\vec{i} + (1/\sqrt{3})\vec{j} + 2\sqrt{2/3}\vec{k}\| = \sqrt{1 + (1/3) + (8/3)} = \sqrt{4} = 2 \\ \|\overrightarrow{BC}\| &= \|-\vec{i} + \sqrt{3}\vec{j}\| = \sqrt{1+3} = 2 \\ \|\overrightarrow{BD}\| &= \|-\vec{i} + (1/\sqrt{3})\vec{j} + 2\sqrt{2/3}\vec{k}\| = \sqrt{1 + (1/3) + (8/3)} = \sqrt{4} = 2 \\ \|\overrightarrow{CD}\| &= \|(1/\sqrt{3} - \sqrt{3})\vec{j} + 2\sqrt{2/3}\vec{k})\| = \sqrt{(1/3 - 2 + 3) + 8/3} = \sqrt{4} = 2 \end{aligned}$$

Thus all the points are 2 units apart.

(b) By solving the equations

$$\begin{aligned} x^2+y^2+z^2 &= (x-2)^2+y^2+z^2 \\ x^2+y^2+z^2 &= (x-1)^2+(y-\sqrt{3})^2+z^2 \\ x^2+y^2+z^2 &= (x-1)^2+(y-1/\sqrt{3})^2+(z-2\sqrt{2/3})^2 \end{aligned}$$

we get $P=(1,1/\sqrt{3},\sqrt{6}/6)$.

(c) The cosine of the angle APB is $1/3$ and the angle is 109.471°.

43. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane containing P, Q, R, and therefore parallel to the normal vector $a\vec{i}+b\vec{j}+c\vec{k}$.

(b)

$$\begin{aligned} \overrightarrow{PQ}\times\overrightarrow{PR} &= (tv-sw-ty+wy+sz-vz)\vec{i}+ \\ &(-tu+rw+tx-wx-rz+uz)\vec{j}+(su-rv-sx+vx+ry-uy)\vec{k} \end{aligned}$$

(c) After substituting $z=(d-ax-by)/c$, $w=(d-au-bv)/c$, $t=(d-ar-bs)/c$ into the result of part (a), and simplifying the expression, we obtain:

$$\begin{aligned} \overrightarrow{PQ}\times\overrightarrow{PR} &= \frac{a(s(u-x)+vx-uy+r(-v+y))}{c}\vec{i}+ \\ &\frac{b(s(u-x)+vx-uy+r(-v+y))}{c}\vec{j}+(s(u-x)+vx-uy+r(-v+y))\vec{k} \\ &= \frac{(s(u-x)+vx-uy+r(-v+y))}{c}(a\vec{i}+b\vec{j}+c\vec{k}). \end{aligned}$$

Thus $\overrightarrow{PQ}\times\overrightarrow{PR}$ is a scalar multiple of $a\vec{i}+b\vec{j}+c\vec{k}$, and hence parallel to it.

CHECK YOUR UNDERSTANDING

1. False. There are exactly two unit vectors: one in the same direction as $\vec{v}$ and the other in the opposite direction. Explicitly, the unit vectors parallel to $\vec{v}$ are $\pm\frac{1}{\|\vec{v}\|}\vec{v}$.

2. False. The length of this vector is $\sqrt{1/3+1/3+4/3}=\sqrt{2}$, not 1.

3. True. Multiplying by a scalar greater than one stretches the length of the vector by the scalar.

4. False. If $\vec{v}$ and $\vec{w}$ are not parallel, the three vectors $\vec{v}$, $\vec{w}$ and $\vec{v}+\vec{w}$ can be thought of as three sides of a triangle. (If the tail of $\vec{w}$ is placed at the head of $\vec{v}$, then $\vec{v}+\vec{w}$ is a vector from the tail of $\vec{v}$ to the head of $\vec{w}$.) The length of one side of a triangle is less than the sum of the lengths of the other two sides. Alternatively, a counterexample is $\vec{v}=\vec{i}$ and $\vec{w}=\vec{j}$. Then $\|\vec{i}+\vec{j}\|=\sqrt{2}$ but $\|\vec{i}\|+\|\vec{j}\|=2$.

5. False. If $\vec{v}$ and $\vec{w}$ are not parallel, the three vectors $\vec{v}$, $\vec{w}$ and $\vec{v}-\vec{w}$ can be thought of as three sides of a triangle. (If the tails of $\vec{v}$ and $\vec{w}$ are placed together, then $\vec{v}-\vec{w}$ is a vector from the head of $\vec{w}$ to the head of $\vec{v}$.) The length of one side of a triangle is less than the sum of the lengths of the other two sides. Alternatively, a counterexample is $\vec{v}=\vec{i}$ and $\vec{w}=\vec{j}$. Then $\|\vec{i}-\vec{j}\|=\sqrt{2}$ but $\|\vec{i}\|-\|\vec{j}\|=0$.

6. False. Two vectors are parallel if and only if one is a nonzero scalar multiple of the other. If $c(\vec{i}-2\vec{j}+\vec{k})=2\vec{i}-\vec{j}+\vec{k}$, then $c=2$ so that the $\vec{i}$ components are equal, but multiplication by 2 does not make the $\vec{j}$ or $\vec{k}$ components equal. Thus, there is no scalar multiple of $\vec{i}-2\vec{j}+\vec{k}$ that is equal to $2\vec{i}-\vec{j}+\vec{k}$.

7. False. As a counterexample, take $3\vec{i}$ and $-\vec{i}$. Then the sum is $2\vec{i}$ which has magnitude 2 (smaller than $\|3\vec{i}\|=3$).

8. False. Since magnitudes are nonnegative this cannot be true when $c<0$. The correct statement is $\|c\vec{v}\|=|c|\|\vec{v}\|$.

9. False. To find the displacement vector *from* $(1,1,1)$ *to* $(1,2,3)$ we subtract $\vec{i}+\vec{j}+\vec{k}$ from $\vec{i}+2\vec{j}+3\vec{k}$ to get $(1-1)\vec{i}+(2-1)\vec{j}+(3-1)\vec{k}=\vec{j}+2\vec{k}$.

10. False. The displacement vector from (a,b) to (c,d) has the same magnitude but opposite direction as the displacement vector from (c,d) to (a,b).

11. False. The dot product is a scalar.

12. True. Components of a normal vector can be read directly from coefficients of x, y and z in the equation for a plane.

13. True. The cosine of the angle between the vectors is negative when the angle is between $\pi/2$ and π.

14. False. The equation $z = x + y$ has normal $\vec{i} + \vec{j} - \vec{k}$, which is not parallel to $\vec{i} + \vec{j} + \vec{k}$. An equation satisfying the given conditions is $x + y + z = 6$.

15. True. The vector from $(0, 1, 0)$ to $(1, 1, 0)$ is $\vec{i}$, while the vector from $(0, 1, 0)$ to $(0, 1, 1)$ is $\vec{k}$, and $\vec{i} \cdot \vec{k} = 0$.

16. True. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, which cannot be negative.

17. False. If the vectors are nonzero and perpendicular, the dot product will be zero (e.g. $\vec{i} \cdot \vec{j} = 0$).

18. False. If $\vec{v}$ and $\vec{w}$ are different vectors, but both are perpendicular to $\vec{u}$, then both $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$ are zero, yet $\vec{v} \neq \vec{w}$. For example, take $\vec{u} = \vec{i}$, $\vec{v} = \vec{j}$ and $\vec{w} = \vec{k}$.

19. True. Using the distributive property, and the fact that $\vec{u} \cdot (-\vec{v}) = -\vec{u} \cdot \vec{v}$, we have

$$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{u} \cdot (-\vec{v}) + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 - \|\vec{v}\|^2.$$

20. True. This vector is $\vec{v}_{\text{Perp}}$, the component of $\vec{v}$ perpendicular to the unit vector $\vec{u}$. To check, calculate the dot product

$$\vec{u} \cdot (\vec{v} - (\vec{v} \cdot \vec{u})\vec{u}) = \vec{u} \cdot \vec{v} - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{u}) = \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} = 0,$$

since $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1$.

21. True. The cross product yields a vector.

22. False. $\vec{u} \times \vec{v}$ has direction *perpendicular* to both $\vec{u}$ and $\vec{v}$.

23. False. This is only true when $\vec{u}$ and $\vec{v}$ are perpendicular. In general, $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$. The value of $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$.

24. True. The left-hand side evaluates to $\vec{k} \cdot \vec{k} = 1$, while the right-hand side evaluates to $\vec{i} \cdot \vec{i} = 1$.

25. False. If $\vec{u}$ and $\vec{w}$ are two different vectors both of which are parallel to $\vec{v}$, then $\vec{v} \times \vec{u} = \vec{v} \times \vec{w} = \vec{0}$, but $\vec{u} \neq \vec{w}$. A counterexample is $\vec{v} = \vec{i}$, $\vec{u} = 2\vec{i}$ and $\vec{w} = 3\vec{i}$.

26. True. Since $(\vec{v} \times \vec{w})$ is perpendicular to $\vec{v}$, the dot product with $\vec{v}$ is zero.

27. True. The cross product is a vector in 3-space, while the dot product is a scalar, so they cannot be equal.

28. True. The cross product $(\vec{i} + \vec{j}) \times (\vec{j} + 2\vec{k}) = 2\vec{i} - 2\vec{j} + \vec{k}$, which has magnitude $\sqrt{2^2 + (-2)^2 + 1^2} = 3$. Since the triangle has area of $1/2$ the parallelogram with the given vectors as sides, the triangle has area $3/2$.

29. True. Any vector $\vec{w}$ that is parallel to $\vec{v}$ will give $\vec{v} \times \vec{w} = \vec{0}$.

30. False. It is not true in general, but there are special cases when $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$. For example, when $\vec{v}$ is parallel to $\vec{w}$, or when one of the vectors is $\vec{0}$. In either case the cross products $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are both the zero vector.

PROJECTS FOR CHAPTER THIRTEEN

1. (a) Let $r = \|\vec{a}\|$ and $s = \|\vec{b}\|$, and let α, β, be the angles between $\vec{a}$, $\vec{b}$, and the x-axis as shown in the figure. Suppose θ is the angle between $\vec{a}$ and $\vec{b}$. We drew the figure with $\alpha < \beta$ and thus $\beta - \alpha = \theta$. If $\alpha > \beta$, then $\alpha - \beta = \theta$. In both cases we know that

$$\text{Area of parallelogram} = \|\vec{a}\|\|\vec{b}\| \sin\theta.$$

Using the formula

$$\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha,$$

and the fact that $a_1 = r\cos\alpha$, $a_2 = r\sin\alpha$, $b_1 = s\cos\beta$, and $b_2 = s\sin\beta$, we get

$$\begin{aligned} a_1b_2 - a_2b_1 &= (r\cos\alpha)(s\sin\beta) - (r\sin\alpha)(s\cos\beta) \\ &= rs(\cos\alpha\sin\beta - \sin\alpha\cos\beta) \\ &= rs\sin(\beta - \alpha) \qquad \text{(from } \sin(\beta-\alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha\text{)} \\ &= \|\vec{a}\|\|\vec{b}\| \sin(\beta - \alpha) \end{aligned}$$

If $\beta > \alpha$, we have $\beta - \alpha = \theta$, so

$$a_1b_2 - a_2b_1 = \|\vec{a}\|\|\vec{b}\| \sin\theta = \text{Area of parallelogram}.$$

If $\beta < \alpha$, we have $\alpha - \beta = \theta$, so

$$|a_1b_2 - a_2b_1| = \|\vec{a}\|\|\vec{b}\||\sin(\beta - \alpha)| = \|\vec{a}\|\|\vec{b}\|\sin\theta = \text{Area of parallelogram.}$$

(b) The sign of $a_1b_2 - a_2b_1$ is the same as the sign of $\beta - \alpha$, so the sign of $a_1b_2 - a_2b_1$ tells us whether the rotation from $\vec{a}$ to $\vec{b}$ is counterclockwise (then $a_1b_2 - a_2b_1$ is positive) or clockwise (then $a_1b_2 - a_2b_1$ is negative).

(c) Part (a) tells us that

$$\text{Area of the parallelogram} = |a_1b_2 - a_2b_1|.$$

The algebraic definition of the cross product is

$$\vec{a} \times \vec{b} = (a_1b_2 - a_2b_1)\vec{k}.$$

The geometric definition has magnitude given by $\|\vec{a} \times \vec{b}\| =$ Area of parallelogram. So the magnitude of the algebraic definition agrees with the magnitude of the geometric definition. To check agreement of the direction of $\vec{a} \times \vec{b}$ for the two definitions, we notice that $(a_1b_2 - a_2b_1)\vec{k}$ is perpendicular to $\vec{a}$ and $\vec{b}$ since $\vec{a}$ and $\vec{b}$ are in the $\vec{i}, \vec{j}$-plane. Also, part (b) says $(a_1b_2 - a_2b_1)\vec{k}$ will point up (down) when the rotation from $\vec{a}$ to $\vec{b}$ is counterclockwise (clockwise). So the direction of the algebraic definition obeys the right-hand rule.

2. (a) We have

$$\|\vec{a}_2\| = \sqrt{0.10^2 + 0.08^2 + 0.12^2 + 0.69^2} = 0.7120$$
$$\|\vec{a}_3\| = \sqrt{0.20^2 + 0.06^2 + 0.06^2 + 0.66^2} = 0.6948$$
$$\|\vec{a}_4\| = \sqrt{0.22^2 + 0.00^2 + 0.20^2 + 0.57^2} = 0.6429$$

$$\vec{a}_2 \cdot \vec{a}_3 = 0.10 \cdot 0.20 + 0.08 \cdot 0.06 + 0.12 \cdot 0.06 + 0.69 \cdot 0.66 = 0.4874$$
$$\vec{a}_3 \cdot \vec{a}_4 = 0.20 \cdot 0.22 + 0.06 \cdot 0.00 + 0.06 \cdot 0.20 + 0.66 \cdot 0.57 = 0.4322$$

The distance between the English and the Bantus is given by θ where

$$\cos\theta = \frac{\vec{a}_2 \cdot \vec{a}_3}{\|\vec{a}_2\|\|\vec{a}_3\|} = \frac{0.4874}{(0.7120)(0.6948)} \approx 0.9852$$

so $\theta \approx 9.9^{\circ}$.

The distance between the English and the Koreans is given by ϕ where

$$\cos\phi = \frac{\vec{a}_3 \cdot \vec{a}_4}{\|\vec{a}_3\|\|\vec{a}_4\|} = \frac{0.4322}{(0.6948)(0.6429)} \approx 0.9676$$

so $\phi \approx 14.6^{\circ}$. Hence the English are genetically closer to the Bantus than to the Koreans.

(b) Let $\vec{a}_5$ be the 4-vector for the half Eskimo, half Bantu population. So

$$\vec{a}_5 = \frac{1}{2}\vec{a}_1 + \frac{1}{2}\vec{a}_2 = (0.195, 0.04, 0.075, 0.68).$$

Then

$$\|\vec{a}_5\| = \sqrt{0.195^2 + 0.04^2 + 0.075^2 + 0.68^2} = 0.7125,$$
$$\vec{a}_3 \cdot \vec{a}_5 = 0.20 \cdot 0.195 + 0.06 \cdot 0.04 + 0.06 \cdot 0.075 + 0.66 \cdot 0.68 = 0.4947.$$

So the distance between the English population and the half Eskimo, half Bantu population is

$$\theta = \arccos\frac{\vec{a}_3 \cdot \vec{a}_5}{\|\vec{a}_3\|\|\vec{a}_5\|} = \arccos\frac{0.4947}{(0.6948)(0.7125)}$$
$$= \arccos 0.9993 \approx 2.1^{\circ}.$$

Since $2.1 < 9.9$, the English are closer to the Bantu/Eskimo mix than to the Bantu alone.

(c) Suppose that x is the fraction of the population that is Eskimo, where $0 \leq x \leq 1$. Then $(1-x)$ is the fraction that is Bantu. (For example, $x = 0.5$, in part (b).) Let $\vec{a}_6$ be the 4-vector for a population that is x Eskimo and $(1-x)$ Bantu. We have

$$\begin{aligned}\vec{a}_6 &= x\vec{a}_1 + (1-x)\vec{a}_2 = \vec{a}_2 + x(\vec{a}_1 - \vec{a}_2) \\ &= (0.10 + 0.19x, 0.08 - 0.08x, 0.12 - 0.09x, 0.69 - 0.02x).\end{aligned}$$

Then

$$\begin{aligned}\|\vec{a}_6\| &= \sqrt{(0.10 + 0.19x)^2 + (0.08 - 0.08x)^2 + (0.12 - 0.09x)^2 + (0.69 - 0.02x)^2} \\ &= \sqrt{0.5069 - 0.024x + 0.051x^2}\end{aligned}$$

and

$$\begin{aligned}\vec{a}_3 \cdot \vec{a}_6 &= 0.20 \cdot (0.10 + 0.19x) + 0.06 \cdot (0.08 - 0.08x) \\ &\quad + 0.06 \cdot (0.12 - 0.09x) + 0.66 \cdot (0.69 - 0.02x) \\ &= 0.4874 + 0.0146x.\end{aligned}$$

Since $\cos\theta$ is a decreasing function of θ for $0 \leq \theta \leq \pi$, to minimize the angle $\theta = \arccos \dfrac{\vec{a}_3 \cdot \vec{a}_6}{\|\vec{a}_3\|\|\vec{a}_6\|}$, we must maximize

$$f(x) = \frac{\vec{a}_3 \cdot \vec{a}_6}{\|\vec{a}_3\|\|\vec{a}_6\|} = \frac{0.4874 + 0.0146x}{0.6948\sqrt{0.5069 - 0.024x + 0.051x^2}}.$$

Using a calculator or computer, we find that the maximum of this function for $0 \leq x \leq 1$ is

$$f(0.5293) = 0.9994.$$

So the minimum distance of $\theta = \arccos(0.9994) = 2.0^\circ$ occurs at a mix of about 52.93% Eskimo and 47.07% Bantu.