BOUNDED ORBIT INJECTIONS AND SUSPENSION EQUIVALENCE FOR MINIMAL $\mathbb{Z}^2$-ACTIONS

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Abstract. In this paper we prove that there exist bounded orbit injections from minimal $\mathbb{Z}^2$ actions of a Cantor set T and S into a common action R if and only if the suspension spaces associated to T and S are homeomorphic. In this way we prove a 2-dimensional analog of a result of Parry-Sullivan on flow equivalence and discrete cross-sections for minimal systems. At the same time the result is a topological analog of a result of del Junco and Rudolph on Kakutani equivalence for ergodic $\mathbb{Z}^d$ actions. We also prove a structural result about such suspension spaces. Namely, that they are a finite union of products of Cantor sets with polygons, $C_i \times P_i$, after an identification on the boundary, $C_i \times \partial P_i$, with the action given by $\mathbb{R}^2$ on the polygon. The polygons P_i can be chosen to have properties associated with Voronoi or Delaunay tilings corresponding to a set of points located uniformly throughout the plane.

1. Introduction

In this paper we prove an equivalence between the existence of orbit preserving injective maps with continuous cocycles between minimal $\mathbb{Z}^d$ Cantor systems and the existence of a homeomorphism between the corresponding suspension spaces for $d = 1, 2$. Let us put this result into some context from both the topological dynamical and ergodic theoretical perspectives.

First we take the topological point of view. In [PS], Parry and Sullivan showed that two homeomorphisms of zero-dimensional sets are flow equivalent if and only if they are discrete cross-sections of a common system. Here we are concerned with a version of this theorem for higher dimensional, free minimal actions, i.e., $\mathbb{Z}^2$ actions of the Cantor set where all orbits are dense. Besides restricting attention to minimal actions, we substitute the two main notions in their statement with those which are more applicable to our setting of $\mathbb{Z}^d$ actions.

First, we substitute the notion of flow equivalence with homeomorphism of the suspension space, or *suspension equivalence*. The difference between these two notions for $\mathbb{Z}$ actions is that in a suspension equivalence one need not preserve the orientation on path components given by a flow. For minimal actions this is nearly a trivial

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consideration since a suspension equivalence must either preserve the orientation on all components or reverse all of them.

Second, we replace the notion of a discrete cross-section with the existence of a bounded orbit injection. We define an orbit injection from one minimal $\mathbb{Z}^d$ action of a Cantor set (X, T) into another (Y, S) to be a 1-1 map $h : X \to Y$ with the property that two points are in the same T orbit if and only if their images under h are in the same S orbit. In particular, there is a map $\beta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$ satisfying

$$S^{\beta(x,v)}h(x) = hT^v(x).$$

To say that h is a bounded orbit injection is to require that β be continuous, or equivalently that $\|\beta(x, v)\|$ is uniformly bounded for $\|v\| = 1$. On the surface, saying that there is a bounded orbit injection from one $\mathbb{Z}$ action (X, T) into another (Y, S) seems weaker than saying that T is a discrete cross-section of S, since the latter condition requires preservation of an order structure, i.e. that $\beta(x, n) > 0$ for $n > 0$. However, if we restrict our attention to minimal actions, a result of Boyle [B] (see also [BT]) on bounded orbit equivalences makes the following formulation of the Parry-Sullivan theorem possible.

Theorem 1. *Two minimal $\mathbb{Z}$ actions S and T on a Cantor set are suspension equivalent if and only if there are bounded orbit injections from S and T into a common $\mathbb{Z}$ action R.*

Our goal in this paper is to prove the above statement for minimal $\mathbb{Z}^d$ actions. The forward direction, that suspension equivalence implies the existence of bounded orbit injections, is not difficult to prove for $\mathbb{Z}^d$ actions. The proof of the reverse direction for $\mathbb{Z}^2$ actions will occupy most of our attention in this paper. A significant step in our proof (in which we use geometric properties of specially constructed Delaunay tilings) requires that we work with $\mathbb{Z}^2$ actions instead of the more general $\mathbb{Z}^d$ actions.

Now let us take a ergodic theoretic point of view. From this perspective, the problem we consider mimics the development of Kakutani equivalence. In [ORW], Ornstein, Rudolph and Weiss proved that if S and T are ergodic transformations which have isomorphic induced maps then S and T are measurable cross-sections to a common ergodic flow. The above equivalence relation in either formulation is called Kakutani equivalence. In [dJR], del Junco and Rudolph proved a higher dimesional version using measure-theoretic orbit injections with cocycles satisfying certain bounds and Katok cross-sections of n-dimensional flows. Thus we present our result also as a topological analog of the result of del Junco and Rudolph.

The proof of the main theorem is a constructive one; beginning with one map, we construct the other. Delaunay tilings of orbits of points play an important role here. We view these as a $\mathbb{Z}^2$ substitute for the Rohlin multitower picture for $\mathbb{Z}$ systems. The core of the proof is taking advantage of the special properties of the bounded orbit injection along with geometric properties of Delaunay tilings to create the map between the suspension spaces.

The outline of the paper is as follows. In the next section, we set out important definitions and give some proofs of simpler statements, leaving only the difficult direction of the main theorem. We outline the proof of the difficult direction at the end of the Section 2.

2. Preliminaries

2.1. **Definitions.** By a Cantor set we will mean a zero-dimensional compact metric space with no isolated points. (All such spaces are homeomorphic.) We consider $\mathbb{Z}^d$ (primarily $d = 2$) acting on a Cantor set X as a group of homeomorphisms. Letting T denote one such action, we denote the action of a vector $v \in \mathbb{Z}^d$ on a point $x \in X$ by $T^v x$. In this paper we will exclusively consider the situation where T is *minimal* and *free*. To say that T is a free $\mathbb{Z}^d$ action is to say that T is aperiodic, or $T^v x = x$ implies $v = 0$. To say that T is minimal is to say that for all $x \in X$, the orbit of x (the set $\{T^v x : v \in \mathbb{Z}^d\}$) is dense in X. When X is a Cantor set and T is a minimal, free $\mathbb{Z}^d$ action of X we will call the pair (X, T) a $\mathbb{Z}^d$ *minimal Cantor system*, or simply, a *minimal Cantor system*. More definitions follow which we give in the setting of minimal Cantor systems, although in most cases identical or similar definitions would work for any free $\mathbb{Z}^d$ action of a Cantor set.

We describe suspension spaces and the relation of suspension equivalence for minimal Cantor systems.

Definition 2. For a $\mathbb{Z}^d$ minimal Cantor system (X, T), let X_T denote the space $X \times \mathbb{R}^d / \sim$ where $(x, u') \sim (y, v')$ if and only if $\exists u, v \in \mathbb{Z}^d$ such that $u' - u = v' - v$ and $T^u x = T^v y$. We call the topological space X_T the *suspension space* for (X, T).

Definition 3. Let (X, T) and (Y, S) be two minimal Cantor systems. We say that (X, T) and (Y, S) are *suspension equivalent* if the suspension spaces X_T and Y_S are homeomorphic.

There is a natural $\mathbb{R}^d$ action on $X \times \mathbb{R}^d$ given by $R^u : (x, v) \mapsto (x, v + u)$. One can check that this action respects the equivalence $\sim$, and thus induces a natural $\mathbb{R}^d$ action on X_T which we will refer to as $\widetilde{T} : X_T \to X_T$.

Next we define bounded orbit injections.

Definition 4. Let (X, T) and (Y, S) be two minimal Cantor systems. An *orbit injection* from (X, T) to (Y, S) is a continuous 1-1 map $h : X \to Y$ such that for all $x, x' \in X$ there is a vector $v \in \mathbb{Z}^d$ such that $T^v x = x'$ if and only if there is a vector $\beta(x, v) \in \mathbb{Z}^d$ such that $S^{\beta(x,v)} h(x) = h(x')$.

We say that an orbit injection h is an *orbit equivalence* if it is onto.

Whether or not the above definition of orbit injection should include the provision that $h(X)$ contain an open set is a debatable point, and this provision seems necessary for more general (non-minimal) versions of this theorem. In our case however, it turns out not to be necessary (see Theorem 7 below). Even more, this additional assumption does not seem to make the proof of our main theorem any simpler in the 2-dimensional case.

Let $\|\cdot\|$ refer to the Euclidean norm on $\mathbb{R}^d$.

Definition 5. We say that an orbit injection h from (X,T) to (Y,S) is *bounded* if there is a number $M>0$ such that for all $v\in\mathbb{Z}^d$, $\|v\|=1$ implies $\|\beta(x,v)\|<M$.

The primary purpose of this paper is to prove the following theorem.

Theorem 6 (Main Theorem). *Let (X,T) and (Y,S) be $\mathbb{Z}^d$ minimal Cantor systems for $d=1,2$. The systems (X,T) and (Y,S) are suspension equivalent if and only if there is a $\mathbb{Z}^d$ minimal Cantor system (Z,R) and bounded orbit injections $h_1:(X,T)\to(Z,R)$ and $h_2:(Y,S)\to(Z,R)$.*

We have the following result, which follows from our main theorem, but for which we do not know a direct proof.

Theorem 7. *Let (X,T) and (Y,S) be $\mathbb{Z}^2$ minimal Cantor systems and suppose the continuous injection $h:X\to Y$ is a bounded orbit injection. Then $h(X)$ is of 2nd category in Y, i.e., $h(X)$ contains an open set.*

We begin by proving the simple cases of Theorem 6.

2.2. **The case when $d=1$.** Let (X,T) and (Y,S) be $\mathbb{Z}$ minimal Cantor systems. To say that T and S are *flow equivalent* is to say that there is an orientation-preserving homeomorphism from X_T to Y_S. Here, the orientation is determined by the natural $\mathbb{R}$ action on the two spaces which extend the $\mathbb{Z}$ actions. Let $\widetilde{x}\in X_T$. The path-connected component of $\widetilde{x}$ is the $\mathbb{R}$ orbit of $\widetilde{x}$ and is dense in X_T.

Suppose $g:X_T\to Y_S$ is a homeomorphism. Then g maps the $\mathbb{R}$ orbit of $\widetilde{x}$ bijectively to the $\mathbb{R}$ orbit of $g(\widetilde{x})$. The map g does so in either an orientation-preserving or orientation-reversing way. But since the $\mathbb{R}$ orbit of $\widetilde{x}$ is dense in X_T and its image is dense in Y_S, g has the same effect on all of the orientations of the $\mathbb{R}$ orbits in the space. In particular, we have the following.

Proposition 8. *Two $\mathbb{Z}$ minimal Cantor systems (X,T) and (Y,S) are suspension equivalent if and only if S is flow equivalent to T or T^{-1}.*

Upon noting the above, the proof of the main theorem for $d=1$ is a corollary of the following theorems.

Theorem 9 ([B]). *Let (X,T) and (Y,S) be $\mathbb{Z}$ minimal Cantor systems. There is a bounded orbit equivalence between T and S if and only if S is conjugate to T or T^{-1}.*

Theorem 10 ([PS]). *Let T and S be $\mathbb{Z}$ actions of a zero-dimensional compact metric space X. Then T and S are flow equivalent if and only if T and S are both conjugate to induced systems of a common $\mathbb{Z}$ action R.*

If R is a $\mathbb{Z}$ action of a zero-dimensional space X, then by an *induced system of* R we mean a clopen set $C \subset X$ along with the first return map $R_C : C \to C$ defined by $R_C(c) = R^n(c)$ where n is the smallest positive integer such that $R^n(c) \in C$. If S is conjugate to an induced system of R and R is minimal, then there is a bounded orbit injection from S into R.

Proof of Theorem 6 for $d = 1$. First suppose that T and S are suspension equivalent. This implies S is flow equivalent to T^e where $e \in \{-1, 1\}$. By the Parry-Sullivan Theorem this implies that T^e and S are both conjugate to induced systems of a common $\mathbb{Z}$ Cantor system (Z, R). Because (Z, R) has induced systems which are minimal, (Z, R) is also minimal. The conjugation maps from S and T^e to induced systems of R are bounded orbit injections from T and S into R.

Conversely, suppose there is a bounded orbit injection h from a $\mathbb{Z}$ minimal Cantor system (X, T) into another (Y, S). Let us presume the additional assumption here that $h(X)$ contains a clopen set $C \subset Y$. (One can show this by modifying the proof of Boyle to our circumstance or by following through the argument in this paper which works equally well for $d = 1, 2$.) Let $D = h^{-1}(C)$ and consider the induced dynamical systems (D, T_D) and (C, S_C). The map $h : D \to C$ is a bounded orbit *equivalence* between T_D and S_C.

Therefore, by Boyle's Theorem, one of T_D, T_D^{-1} is conjugate to S_C. By the Parry-Sullivan Theorem, this happens if and only if T or T^{-1} is flow equivalent to S. Therefore, T and S are suspension equivalent.

Similarly, if bounded orbit injections exist from T and S into a third system R then since suspension equivalence is an equivalence relation, T and S are suspension equivalent. □

2.3. **One direction of the main theorem when** $d > 1$**.** Here we provide a proof of the simpler direction of the main theorem for any $d \geq 2$.

Proof of $\Rightarrow$ *in Theorem 6.* We begin by noting that given a minimal $\mathbb{Z}^d$ action T there is a simple way to construct examples of a minimal action T' with a bounded orbit injection from T into T', by constructing a "tower of size m over T". Fix any $m \geq 0$ and let $Q_m = \{0, 1, 2, \ldots, m-1\}^d$ with the discrete topology. Let $X_m = X \times Q_m$, a Cantor set. For $(x, q) \in X_m$, and $u \in \mathbb{Z}^d$, we can write $u + q = mv + r$ where $m \in \mathbb{Z}$, $v \in \mathbb{Z}^d$ and $r \in Q_m$. Define an action T_m on X_m by $(T_m)^u(x, q) = (T^v x, r)$. There is a bounded orbit injection from T into T_m defined by $x \mapsto (x, 0)$. Note that there is a natural homeomorphism from X_T into $(X_m)_{T_m}$ given by $g_m : (x, u) \mapsto ((x, 0), mu)$.

Now assume that g is a homeomorphism from X_T to Y_S. Because the $\mathbb{R}^d$ orbits of points in the suspension spaces are precisely the path-connected components of the space, we have that g maps each $\widetilde{T}$ orbit onto an $\widetilde{S}$ orbit. Thus, associated to g there must be an $\mathbb{R}^d$ cocycle $\alpha : X_T \times \mathbb{R}^d \to \mathbb{R}^d$ such that for $(\widetilde{x}, u) \in X_T \times \mathbb{R}^d$,

$$g(\widetilde{T}^u(\widetilde{x})) = \widetilde{S}^{\alpha(\widetilde{x}, u)} g(\widetilde{x}).$$

We now wish to prove a statement that mimics boundedness for a bounded orbit injection. A statement like, there exists $K > 0$ such that $\|\alpha(\widetilde{x}, v)\| < K\|v\|$ for any $v \in \mathbb{R}^2$ and $\widetilde{x} \in X$, except that in general, it is doesn't appear that such a statement needs to be true (α is continuous, but it is not clear it needs to be Lipschitzian). Regardless, such a statement is true where we want it to be true, for v bounded away from zero.

The continuity of g implies that $\lim_{\|v\|\to 0} \alpha(\widetilde{x}, v) = 0$. So, given $\epsilon > 0$ there exists δ such that $\|v\| < \delta$ implies $\|\alpha(\widetilde{x}, v)\| < \epsilon$. The compactness of X_T implies that δ can be chosen uniformly with respect to $\widetilde{x}$. Moreover, we may assume $0 < \delta \leq 1/2$. Take $K = \frac{3}{2}\frac{\epsilon}{\delta}$. For v with $\|v\| \geq 1$ we can write $v = \sum_{i=1}^{n+1} v_i$ where $n \geq 2$, $\gamma \in \mathbb{R}$ is such that $v_i = \gamma v$, with $\|v_i\| = \delta$ for $i = 1, .., n$ and $\|v_{n+1}\| \leq \delta$. This implies that $n\delta \leq \|v\| \leq (n+1)\delta$. Applying the cocycle equation

$$\alpha\left(\widetilde{x}, u+v\right) = \alpha\left(\widetilde{x}, u\right) + \alpha\left(\widetilde{T}^u \widetilde{x}, v\right)$$

we have (with $v_0 = 0$)

$$\|\alpha(\widetilde{x}, v)\| \leq \sum_{i=1}^{n+1} \|\alpha(\widetilde{T}^{v_{i-1}}\widetilde{x}, v_i)\| \leq (n+1)\epsilon = \delta \cdot n \cdot \frac{(n+1)}{n}\frac{\epsilon}{\delta} \leq \|v\| \cdot K$$

The important fact which we extract from the preceding argument is that if $\alpha\left(\widetilde{x}, v\right) \in \mathbb{Z}^d \setminus \{0\}$ then $\|v\| > 1/K$.

By partitioning the Cantor space X_m into clopen sets $\{Z_i\}$ of sufficiently small diameter, we can guarantee that $z, z' \in Z_i$ and $(T_m)^v(z) = z'$ for $\|v\| \leq 2$ implies $z = z'$. Also, note that the sets $P_i = \left\{\left(\widetilde{T_m}\right)^v (Z_i \times \{0\}) : \|v\| < \frac{2}{3}\sqrt{d}\right\}$ form an open cover of the suspension space $(X_m)_{T_m}$. The composition of maps $g_m g^{-1}$ is a homeomorphism from Y_S to $(X_m)_{T_m}$. By the conclusion of the previous paragraph, for $v \in \mathbb{Z}^2 \setminus \{0\}$ the preimages $g^{-1}(y, v)$ and $g^{-1}(v, 0)$ are more than $1/K$ from each other (*orbitally*). So, $g_m g^{-1}(y, v)$ and $g_m g^{-1}(y, 0)$ are more than m/K from each other (orbitally).

Select $m > 2\sqrt{d}K$. So $g_m g^{-1}(y, v)$ and $g_m g^{-1}(y, 0)$ are more than $2\sqrt{d}$ from each other (orbitally). So, each set $\left\{\left(\widetilde{T_m}\right)^v (z, 0) : \|v\| < \frac{2}{3}\sqrt{d}\right\}$ contains at most one point of the form $g_m g^{-1}(y, 0)$ where $y \in Y$. Using the Lebesgue Lemma, if Y is partitioned into disjoint clopen sets $\{D_j\}$ of sufficiently small diameter, for each j there is an i such that $g^{-1}(D_j \times \{0\}) \subset P_i$. Finally, let $y \in D_j$. Define $h(y)$ as the unique point $z \in Z_i \subset X_m$ such that $g^{-1}(y, 0) = \left(\widetilde{T_m}\right)^v (z, 0)$ for $\|v\| < \frac{2}{3}\sqrt{d}$. The map h is a bounded orbit injection from S into T_m. $\square$

The next five sections of the paper are dedicated to proving the theorem below, the converse direction of the main theorem when $d = 2$.

Theorem 11. *Let (X,T) and (Y,S) be $\mathbb{Z}^2$ minimal Cantor systems and suppose there is a bounded orbit injection from (X,T) to (Y,S). Then (X,T) and (Y,S) are suspension equivalent.*

Here we give a brief outline of the proof. In Section 3, we establish some important properties of bounded orbit injections and define a continuous map $g : X_T \to Y_S$ which is an injection of $\mathbb{R}^2$ orbits. The main issue to resolve is that g is not necessarily 1-1 on X_T.

Note that the $\mathbb{R}^2$ orbit of a point $\widetilde{x} \in X_T$ is the continuous, 1-1 image of $\mathbb{R}^2$. Through this correspondence, we define tilings of X_T. Roughly, a tiling of X_T is a continuous, translation-commuting assignment of a family of tilings of $\mathbb{R}^2$ to points in X_T. The tilings we use are Delaunay tilings with some special properties. All of this is made precise in Section 4, the main lemma of which is the Tiling Lemma 22.

Suppose there is a bounded orbit injection from T to S with corresponding cocycle function $\beta : X \times \mathbb{Z}^2 \to \mathbb{Z}^2$. We can then extend β to a map $\widetilde{\beta} : X \times \mathbb{R}^2 \to \mathbb{R}^2$ linearly. Fixing $\widetilde{x} \in X_T$, we have a map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\phi(v) = \widetilde{\beta}(\widetilde{x}, v)$. By restricting ϕ to the union of the boundaries of the tiles in the tiling corresponding to $\widetilde{x}$, we have a continuous map $\partial\phi$ on a planar graph. The heart of the proof is to establish the Graph Isomorphism Lemma in Section 5. In this section, we show that the properties of the bounded orbit injection, along with the geometric properties of the Delaunay tilings are strong enough to repair the lack of injectivity of $\partial\phi$ on the planar graph to obtain a graph isomorphism $\partial\psi$. Then it is a simple matter to extend the graph isomorphism $\partial\psi$ to a homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$. By taking care along the way, we can insure that $\widetilde{T}^v(\widetilde{x}) \mapsto \widetilde{S}^{\psi(v)} g(\widetilde{x})$ is a continuous map which can be extended to a homeomorphism from X_T and Y_S. The details of the extension argument are laid out in Section 6.

3. Bounded Orbit Injections

Assume that h is a bounded orbit injection from one $\mathbb{Z}^2$ minimal Cantor system (X,T) to another $\mathbb{Z}^2$ minimal Cantor system (Y,S). Let β be as in Definition 4. The function β satisfies the following cocycle equation for any $x \in X$ and $v, w \in \mathbb{Z}^2$.

$$\beta(x, v+w) = \beta(x,v) + \beta(T^v x, w) \tag{3.1}$$

That h is bounded means for some M, $\|v\| = 1$ implies $\|\beta(x,v)\| < M$.

Proposition 12. *For any $v \in \mathbb{Z}^2$, $\|\beta(x,v)\| < 2\|v\|M$.*

Proof. Let $0 = w_0, w_1, w_2, \ldots, w_n = v$ be a sequence of vectors in $\mathbb{Z}^2$ such that $\|w_{i+1} - w_i\| = 1$ and $n < 2\|v\|$. Then since $\beta(x,v) = \sum_{i=0}^{n-1} \beta(T^{w_i} x, w_{i+1})$, we have $\|\beta(x,v)\| < Mn < 2\|v\|M$. □

For each $u, v \in \mathbb{Z}^2$, let $E_u^v = \{x \in X : \beta(x,v) = u\}$.

Proposition 13. *For any $v, u \in \mathbb{Z}^2$, E_u^v is clopen.*

Proof. Suppose that $\{x_n\}$ is a sequence of points in E_u^v with $\lim_{n\to\infty} x_n = x$. Then we have for all n, $h(T^v x_n) = S^u h(x_n)$. Since h, T^v and S^u are all continuous, taking $\lim_{n\to\infty}$ of both sides yields $h(T^v x) = S^u h(x)$, which implies $x \in E_u^v$. Therefore E_u^v is closed. By the previous proposition, for fixed v, there can only be finitely many u for which E_u^v is nonempty. Since $X = \bigcup_u E_u^v$ each set E_u^v is clopen. □

Corollary 13.1. *The function $\beta : X \times \mathbb{Z}^2 \to \mathbb{Z}^2$ is continuous.*

What follows next is a key observation about bounded orbit injections. From the definition we have an upper bound on $\|\beta(x,v)\|$ depending on $\|v\|$, and this also gives a lower bound. Let $E_u = \cup_{v\in\mathbb{Z}^2} E_u^v$.

Proposition 14. *For any $u \in \mathbb{Z}^2$, E_u is clopen.*

Proof. Fix $u \in \mathbb{Z}^2$. The set E_u is clearly open, so we need to show it is closed. Take a sequence $x_n \in E_u$ which converges to $x \in X$. Because S^u and h are continuous, $\lim_{n\to\infty} S^u h(x_n) = S^u h(x)$. On the other hand each $x_n \in E_u^{v_n}$ for some unique v_n. There exists a subsequence $T^{v_{n(k)}} x_{n(k)}$ which converges, to say z. So $\lim_{n(k)\to\infty} h(T^{v_{n(k)}}(x_{n(k)})) = h(z)$. But $h(T^{v_{n(k)}}(x_{n(k)})) = S^u h(x_{n(k)}) \to S^u h(x)$. So $h(z) = S^u h(x)$, thus $S^u h(x) \in h(X)$. Because h is an orbit injection there exists $v \in \mathbb{Z}^2$ such that $z = T^v x$. Now since $h(T^v x) = S^u h(x)$, $x \in E_u^v \subset E_u$. So E_u is closed. □

Corollary 14.1. *For any $M > 0$ there is a number $N > 0$ such that for all $x \in X$ and $v \in \mathbb{Z}^2$, if $\|v\| > N$ then $\|\beta(x,v)\| > M$.*

Proof. By Proposition 14, there exists an N_u such that for all $x \in X$ and $v \in \mathbb{Z}^2$, if $\|v\| > N_u$ then $\beta(x,v) \neq u$, because E_u^v is empty for v sufficiently large. So the same is true for the finite collection of $u \in \mathbb{Z}^2$ with $\|u\| \leq M$. □

3.1. **Constructing a map from X_T to Y_S.** Recall that X_T is the space $X \times \mathbb{R}^2$ modulo the relation $(x,u') \sim (y,v')$, if and only if $\exists u, v \in \mathbb{Z}^d$ such that $u' - u = v' - v$ and $T^u x = T^v y$. Let $\Pi_T : X \times \mathbb{R}^2 \to X_T$ denote the quotient map. We define the $\mathbb{R}^2$ *flow* action R on $X \times \mathbb{R}^2$ by $R^t(x,r) = (x, r+t)$ for $t \in \mathbb{R}^2$. Note that the flow action R respects the equivalence relation and thus projects down to induce the flow action $\widetilde{T}$ on X_T, where $\widetilde{T}^v \Pi_T = \Pi_T R^v$ for all $v \in \mathbb{R}^2$. Because $T^v x = x$ implies $v = 0$ (T is free), there is a bijection between $\mathbb{R}^2$ and each $\widetilde{T}$ orbit.

Observe, if a function $f : X \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies an equation similar to Equation 3.1,

$$f(x, v+w) = \beta(x,v) + f(T^v x, w)$$

(which we call a *β-mediated cocycle* equation), then there is a well-defined function $\tilde{f} : X_T \to Y_S$ defined by $\tilde{f}(\widetilde{x}) = \Pi_S(h(x), f(x,u))$, for any $(x,u) \in \Pi_T^{-1}(\widetilde{x})$. This idea will be used several times beginning with the following.

Given the bounded orbit injection $h : X \to Y$, we now construct a continuous map $g : X_T \to Y_S$ by (essentially) extending the map h linearly. We do this as follows (see

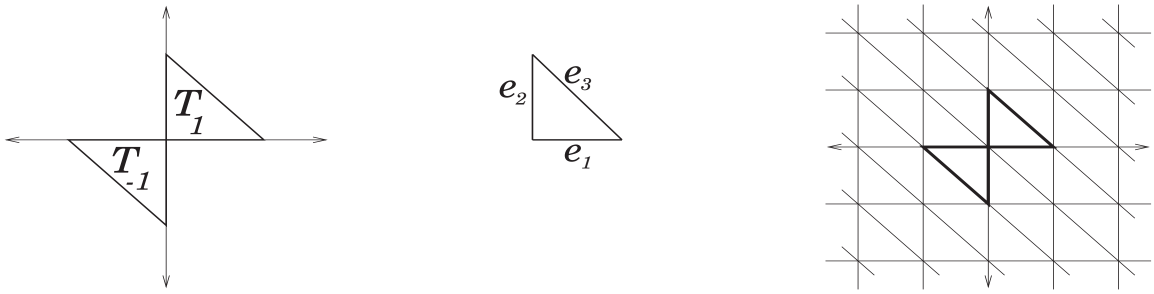


FIGURE 1. Extending h with a graph.

Figure 1). Let G be the graph with vertex set $\mathcal{V} = \mathbb{Z}^2$, and edge set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where $\mathcal{E}_i = \{v + te_i : v \in \mathbb{Z}^2, t \in [0,1]\}$ for $i = 1,2$ and $\mathcal{E}_3 = \{v + t(e_1 - e_2) : v \in \mathbb{Z}^2, t \in [0,1]\}$. Let T_1 (and T_{-1}) be the triangles formed by the closed convex hull(s) of 0, e_1, e_2 (and 0, $-e_1$, $-e_2$).

For each $x \in X$, the map $v \mapsto \beta(x,v)$ is defined on $\mathbb{Z}^2$. We can extend this map to the edges of the triangles and their interiors linearly. That is, if $p \in \mathbb{R}^2$ then p is a convex combination of vertices v, w, u of (at least) one of the triangles, i.e., $p = \alpha_1 v + \alpha_2 w + \alpha_3 u$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$. We then define $\Gamma(x,p) = \alpha_1\beta(x,v) + \alpha_2\beta(x,w) + \alpha_3\beta(x,u)$. In addition to being a well-defined map from $X \times \mathbb{R}^2 \to \mathbb{R}^2$, Γ satisfies a β mediated cocycle equation,

$$\Gamma(x, r + v) = \beta(x,v) + \Gamma(T^v x, r)$$

for all $x \in X$, $v \in \mathbb{Z}^2$, and $r \in \mathbb{R}^2$. Moreover, analogs of propositions 12, 13, and 14.1 and their corollaries exist. For the most part, we leave the proofs to the reader.

Proposition 12′. *For any $v \in \mathbb{R}^2$, $\|\Gamma(x,v)\| \leq 2\|v\|M$.*

Proposition 13′. *There exists a clopen partition $\{X_j\}$ of X such that $\beta(x,0)$, $\beta(x,\pm e_1)$, $\beta(x,\pm e_2)$ are constant on each X_j. And hence the images $\Gamma(x,t)$ for $t \in T_i$ and $i = \pm 1$ are constant on each X_j, as well.*

Corollary 13.1′. *The function $\Gamma : X \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.*

Corollary 14.1′. *Given $M > 0$ there is an $N > 0$ such that for any $u, v \in \mathbb{R}^2$ with $\|u - v\| > N$ and for any $x \in X$, $\|\Gamma(x,u) - \Gamma(x,v)\| > M$.*

Proof. Fix $x \in X$, and let u be inside the triangle with vertices u_1, u_2 and u_3 and v inside the triangle with vertices v_1, v_2 and v_3. Corollary 14.1 shows that if $\|v_i - u_j\|$ is sufficiently large, then so is $\|\Gamma(x,v_i) - \Gamma(x,v_j)\|$ for all $i, j \in \{1,2,3\}$. Now it follows from Proposition 12 that any point in convex hull of the vertices $\Gamma(x,u_1), \Gamma(x,u_2), \Gamma(x,u_3)$ is far away from any point in the convex hull of $\Gamma(x,v_1), \Gamma(x,v_2), \Gamma(x,v_3)$. All bounds are uniform in x since this was true in Propositions 12 and Corollary 14.1. □

As mentioned earlier, because Γ satisfies a β-mediated cocycle equation there is well-defined map $g : X_T \to Y_S$ defined as follows. Let

$$g(\widetilde{x}) = \Pi_S(h(x), \Gamma(x,r)).$$

where $(x, r) \in \Pi_T^{-1}(\widetilde{x})$. The central difficulty here is that one can not expect g to be injective. It is the purpose of this paper to show that there is a perturbation of g which is injective.

Let $\widetilde{T}$ and $\widetilde{S}$ denote the $\mathbb{R}^2$ actions on X_T and Y_S, respectively. While g is not an injection, g is an injection on the orbits of these $\mathbb{R}^2$ actions. That is, $\widetilde{x}$ and $\widetilde{x}'$ are in the same $\widetilde{T}$ orbit if and only if the images $g(\widetilde{x})$ and $g(\widetilde{x}')$ are in the same $\widetilde{S}$ orbit. Thus, associated to g there must be an $\mathbb{R}^2$ cocycle $\alpha : X_T \times \mathbb{R}^2 \to \mathbb{R}^2$ such that for $(\widetilde{x}, r) \in X_T \times \mathbb{R}^2$,

$$g(\widetilde{T}^r(\widetilde{x})) = \widetilde{S}^{\alpha(\widetilde{x},r)} g(\widetilde{x}).$$

We can also calculate α in terms of Γ for any $(x, s) \in \Pi_T^{-1}(\widetilde{x})$ as follows

$$\alpha(\widetilde{x}, r) = \Gamma(x, s + r) - \Gamma(x, s).$$

The cocycles α and β share some properties. For example, for $\widetilde{x} = \Pi_T(x, 0)$ and $v \in \mathbb{Z}^2$, $\alpha(\widetilde{x}, v) = \beta(x, v)$, and again we have a series of statements.

Proposition 12″. *For any $r \in \mathbb{R}^2$ and $\widetilde{x} \in X_T$, $\|\alpha(\widetilde{x}, r)\| \leq 2\|r\|M$.*

Let $A = \Pi_T(X \times \mathbb{Z}^2)$. In fact, A is homeomorphic to X because $\Pi_T(X \times \mathbb{Z}^2) = \Pi_T(X \times \{0\})$ and Π_T is injective on the latter. For any $\widetilde{x} \in X_T$ there exist $\widetilde{a} \in A$ and $r \in T_1 \cup T_2$ such that $\widetilde{T}^r \widetilde{a} = \widetilde{x}$ (because this holds above in $X \times \mathbb{R}^2$).

Proposition 13″. *Given $n > 0$, there exists a clopen partition $\{A_j\}$ of A such that if $\widetilde{x}, \widetilde{y} \in A_j$ and $\|r\| < n$ then $\alpha(\widetilde{x}, r) = \alpha(\widetilde{y}, r)$.*

Corollary 13.1″. *The function $\alpha : X_T \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.*

Corollary 14.1″. *Given $M > 0$ there is an $N_M > 0$ such that for any $u, v \in \mathbb{R}^2$ with $\|u - v\| > N_M$ and for any $\widetilde{x} \in X_T$, $\|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > M$.*

The construction in this section gives us a better sense of the method of proof. As we have done here, we will construct a tiling of X_T by action polygons. The polygons here are all triangles, but in the next section, they will be polygons associated with a Delanay tiling. The map g constructed in this section gives a well-defined map ∂g from the graph defined by the boundaries of the Delanay tiles to Y_S. The map ∂g is not 1-1, but the properties described in this section will be strong enough to allow us to repair ∂g to a graph isomorphism. Then we will be just left with the relatively simple matter of extending ∂g to all of X_T.

4. Suspension space tilings

There is a great deal of literature on Voronoi and Delaunay tilings (e.g. [A, OBS] for introductions and physical science applications, [L1, L2, P] for some recent applications to dynamics relevant to us). We will review some of the essentials for our construction.

4.1. **Tilings of $\mathbb{R}^2$ by polygons.** By *a tiling of* $\mathbb{R}^2$ *by convex polygons* we mean a countable collection $\mathcal{D}$ of convex polygons which cover $\mathbb{R}^2$, have disjoint interiors, and meet edge to edge. Define the *0-dimensional faces* of a convex polygon to be the extreme points of the polygon. Define the *1-dimensional faces* of a convex polygon to be the closed line segments in the polygon boundary connecting the extreme points. The set of faces of a convex polygon is the union of the 0-dimensional and the 1-dimensional faces of the polygon. For us, tilings have the property that the (non-empty) intersection of any two distinct polygons in the tiling is a face in both polygons.

A tiling of $\mathbb{R}^2$ by polygons is *finite* if the set of polygons up to equivalence by translation only consists of a finite number of polygons. So, there exists a list $\{P_i\}_{i=1}^K$ of *prototiles* such that for each $D \in \mathcal{D}$, $D = P_i + v$ for some $i \in 1, .., K$ and $v \in \mathbb{R}^2$. A *full tiling space for* $\{P_i\}_{i=1}^K$ is the set of all finite tilings of $\mathbb{R}^2$ which can be made with prototiles $\{P_i\}_{i=1}^K$. This space is compact with the appropriate topology, and $\mathbb{R}^2$ acts on the elements of the full tiling space by translation, $\sigma^v\mathcal{D} = \mathcal{D} - v$.

4.2. **Tiling the boundary of a tiling with a graph.** For any finite tiling of $\mathbb{R}^2$ by convex polygons $\mathcal{D}$, there is an associated boundary $\partial\mathcal{D} = \cup\{\partial D : D \in \mathcal{D}\}$. Each prototile P_i is the convex hull of a finite set of vertices, and each tile D in a finite tiling is the convex hull of a translate of these vertices. Let $\mathcal{V}(D)$ be the vertices of D and let $\mathcal{E}(D)$ be the set of line segments that connect the vertices of D without intersecting the interior of D. That is, $\mathcal{E}(D)$ is the set of line segments forming the boundary of D. We associate to the tiling $\mathcal{D}$ the graph $(\mathcal{V}, \mathcal{E})$ whose vertex set $\mathcal{V} = \cup\{\mathcal{V}(D) : D \in \mathcal{D}\}$ and whose (undirected) edge set $\mathcal{E} = \cup\{\mathcal{E}(D) : D \in \mathcal{D}\}$ is the union of its polygons' edges. So $\partial\mathcal{D} = \cup\{e : e \in \mathcal{E}\}$. We will use the notation $[v, w]$ (or $[w, v]$) to denote the edge with endpoints v and w. For each $v \in \mathcal{V}$ there is a subset $\mathcal{E}_v \subset \mathcal{E}$ of edges which have v as an endpoint ($\mathcal{E}_v = \{e \in \mathcal{E} : v \in e\}$). Let $star(v) = \cup\{e : e \in \mathcal{E}_v\}$ (the star of v in the usual sense in the graph $\partial\mathcal{D}$).

4.3. **Voronoi and Delaunay Tilings of $\mathbb{R}^2$.** Let $\mathcal{V}$ be a subset of $\mathbb{Z}^2$. A subset $\mathcal{V} \subset \mathbb{Z}^2$ is said to be *m-separated* if for every pair $u, v \in \mathcal{V}$ of distinct elements, $\|u - v\| > m$. We say $\mathcal{V} \subset \mathbb{Z}^2$ is *m-syndetic* in $\mathbb{Z}^2$ if for every $u \in \mathbb{Z}^2$ there exists $v \in \mathcal{V}$ such that $\|u - v\| \leq m$. If $\mathcal{V} \subset \mathbb{Z}^2$ is both m-separated and m-syndetic in $\mathbb{Z}^2$ then we will say $\mathcal{V}$ is *m-regular* in $\mathbb{Z}^2$. Let $\mathfrak{M}_m = \{\mathcal{V} \subset \mathbb{Z}^2 : \mathcal{V}$ is m-regular in $\mathbb{Z}^2\}$. We will construct tilings based on m-regular subsets of $\mathbb{Z}^2$. For $u \in \mathbb{R}^2$ and $n > 0$, let $B(u, n)$ denote the ball in $\mathbb{R}^2$ centered at u with radius n with respect to the distance d, the Euclidean metric ($d(u, v) = \|u - v\|$). Similarly, let $\overline{B}(u, n)$ be the closure of $B(u, n)$ in $\mathbb{R}^2$ and $\partial B(u, n) = \partial\overline{B}(u, n)$ be the boundary of $B(u, n)$ in $\mathbb{R}^2$.

For $v \in \mathcal{V} \subset \mathbb{R}^2$ the *Voronoi tile* corresponding to v with respect to $\mathcal{V}$ is the set $V_v \equiv \{r \in \mathbb{R}^2 : d(r, v) \leq d(r, \mathcal{V})\}$. Let $\mathfrak{V}(\mathcal{V}) = \{V_v : v \in \mathcal{V}\}$. Fix $m \geq 1$ and suppose that $\mathcal{V} \subset \mathbb{Z}^2$ and $\mathcal{V}$ is m-regular. Then the collection $\mathcal{P} = \{V_v - v : v \in \mathcal{V}$ for some $\mathcal{V} \in \mathfrak{M}_m\}$ is a finite set of convex polygonal prototiles $\mathcal{P} = \{P_i\}_{i=1}^K$ and $\overline{B}(0, m/2) \subset P_i \subset \overline{B}(0, m+1)$ for all i [L1]. Thus the collection $\mathfrak{V}(\mathcal{V})$ is a finite tiling

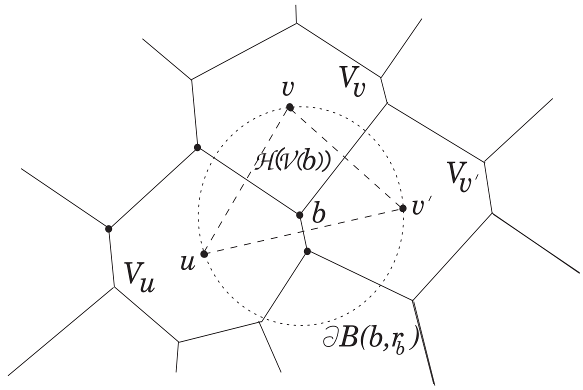


FIGURE 2. Voronoi and Delaunay tiles: $u, v, w \in \mathcal{V}$ and $b \in \widehat{\mathcal{V}}$

of $\mathbb{R}^2$ which we refer to as the *Voronoi tiling* corresponding to $\mathcal{V}$. For $m \geq 1$ fixed, the set of all tilings of the form $\mathfrak{V}(\mathcal{V})$ where $\mathcal{V} \in \mathfrak{M}_m$ is a closed shift-invariant subset of the full tiling space for the set of prototiles $\{P_i\}_{i=1}^K$.

To construct the Delaunay tiling corresponding to $\mathcal{V}$, for each $p \in \mathbb{R}^2$ we define $\mathcal{V}(p) = \{v \in \mathcal{V} : d(p, v) = d(p, \mathcal{V})\}$ and let $r_p = d(p, \mathcal{V})$ (thus, $\mathcal{V}(p) \subset \partial B(p, r_p)$ and $\mathcal{V}(p) \cap B(p, r_p) = \emptyset$). (See Figure 2.) That is, $\mathcal{V}(p)$ is the set of elements in $\mathcal{V}$ nearest to p and equidistant from p. We let $\mathcal{H}(\mathcal{V}(p))$ denote the closed convex hull of $\mathcal{V}(p)$, and we let $\widehat{\mathcal{V}} = \{p \in \mathbb{R}^2 : |\mathcal{V}(p)| \geq 3\}$, where $|E|$ denotes the cardinality of the set E. Then, for $p \in \widehat{\mathcal{V}}$, the set $\mathcal{H}(\mathcal{V}(p))$ is a convex polygon inscribed in the circle $\partial B(p, r_p)$. We refer to $\mathcal{H}(\mathcal{V}(p))$ as the *Delaunay tile* corresponding to p with respect to $\mathcal{V}$. Some may be interested to note that p need not be an element of $\mathcal{H}(\mathcal{V}(p))$. The collection $\mathfrak{D}(\mathcal{V}) = \{\mathcal{H}(\mathcal{V}(p)) : p \in \widehat{\mathcal{V}}\}$ is a tiling of $\mathbb{R}^2$ known as the *Delaunay tiling of* $\mathbb{R}^2$ *corresponding to* $\mathcal{V}$.

Because $\mathcal{H}(\mathcal{V}(p))$ is determined by a set of elements of $\mathbb{Z}^2$ that are within $m + 1$ of p, there are only a finite number of such sets possible (for fixed $m \geq 1$). So the set $\{q \in [0, 1) \times [0, 1) : q \in \widehat{\mathcal{V}}$ for some $\mathcal{V} \in \mathfrak{M}_m\}$ is finite. Consequently, $\{\mathcal{H}(\mathcal{V}(p)) - p : p \in \widehat{\mathcal{V}}$ for some $\mathcal{V} \in \mathfrak{M}_m\}$ is a finite set of convex polygonal prototiles. The tilings $\mathfrak{V}(\mathcal{V})$ and $\mathfrak{D}(\mathcal{V})$ are said to be dual to each other, and we will refer to $\mathfrak{D}(\mathcal{V})$ as the Delaunay tiling associated to the Voronoi tiling $\mathfrak{V}(\mathcal{V})$, and vise versa.

We will work with the graph $G(\mathcal{V})$ associated with the boundary of the Delaunay tilings $\mathfrak{D}(\mathcal{V})$. A consequence of the above construction is that the vertex set of $G(\mathcal{V})$ is simply $\mathcal{V}$ and the edge set $\mathcal{E}$ is comprised of the collection of line segments connecting vertices which are adjacent on the circle $\partial B(p, r_p)$ for $p \in \widehat{\mathcal{V}}$. Again, see Figure 2.

Both the Voronoi tilings and the Delaunay tilings have graphs associated to them, however we will only ever work with the graph associated to the Delaunay tiling. Nonetheless, to clarify that we are referring to edges and vertices in the graph associated to the Delaunay tiling, we use the terms Delaunay edges and Delaunay vertices.

The following facts are important to our construction (see [L2] for details).

Fact 15. *There is a $\delta > 0$ such that for any $m \geq 1$, if $\mathcal{V}$ is an m-regular set then the minimum angle formed by two Delaunay edges meeting at a Delaunay vertex is at least δ.*

Fact 16. *(Cor 6.4.2 [L2]) Given $K > 0$ there exists $m_K \geq 2K$ such that for $m \geq m_K$, if $\mathcal{V}$ is m-regular then for any pair of Delaunay edges $l, l' \in \mathcal{E}$, either $l \cap l' \neq \emptyset$ or $d(l, l') > 2K$.*

Remark 17. Fact 16 fails to hold for d-dimensional Delaunay tilings corresponding to m-regular vertex sets for $d > 2$. This property will be critical for our construction and represents the main obstruction to generalizing our results to $\mathbb{Z}^d$ actions for $d > 2$. We explain further in Remark 23 and Section 7.

Remark 18. For $m \geq m_K$, if $\mathcal{V}$ is m-regular then $\mathcal{V}$ is a $2K$ separated subset of $\mathbb{Z}^2$, and (in particular) every Delaunay edge $l \in \mathcal{E}$ has a length of $2K$ or greater.

4.4. **Remarks about Π_T and geometry in X_T orbits.** We will often consider Π_T restricted to a single path-connected component $\{x\} \times \mathbb{R}^2$. This restriction is a continuous bijection which respects geometry in the following sense. Suppose $l \subset \mathbb{R}^2$ is a straight line, and consider the set $\Pi_T(x, l)$. Because $\Pi_T^{-1}(\Pi_T(x, l)) = \cup_{v \in \mathbb{Z}^2}(T^v x, l - v)$, it follows that if any subset $l' \subset \mathbb{R}^2$ and $x' \in X$ are such that $\Pi_T(x', l') = \Pi_T(x, l)$, then l' is also a straight line (in fact, $l' = l - v$ for the unique $v \in \mathbb{Z}^2$ such that $x' = T^v x$). Thus, we may refer to the set $\Pi_T(x, l) \subset X_T$ unambiguously as a straight line. Similarly, a set $L \subset \{\widetilde{T}^v \widetilde{x} : v \in \mathbb{R}^2\}$ (for $\widetilde{x} \in X_T$) may be said to be a line segment if and only if there exists a line segment l in $\mathbb{R}^2$ such that $L = \{\widetilde{T}^v \widetilde{x} : v \in l\}$. In this sense many aspects of Euclidean geometry in $\mathbb{R}^2$ exist in the $\widetilde{T}$ orbits of X_T. We will use this property to speak without ambiguity about certain tilings and their associated graphs in the $\widetilde{T}$ orbits of X_T, as well as such aspects as the angle between two edges occurring in those graphs.

In this spirit it will be convenient for us to define an *orbital metric* $d_T : X_T \times X_T \to \mathbb{R}^* = \mathbb{R} \cup \infty$ (and d_S on $Y_S \times Y_S$). For $\widetilde{x}, \widetilde{y} \in X_T$, if $\widetilde{x}$ and $\widetilde{y}$ are not in the same $\widetilde{T}$ orbit, then define $d_T(\widetilde{x}, \widetilde{y}) = \infty$. If $\widetilde{x}$ and $\widetilde{y}$ are in the same $\widetilde{T}$ orbit, then $\widetilde{x} = \widetilde{T}^v \widetilde{y}$ for some $v \in \mathbb{R}^2$ (because (X, T) is free, v is unique), we define $d_T(\widetilde{x}, \widetilde{y}) = \|v\|$, where $\|v\|$ is the Euclidean norm of v. For $E, F \subset X_T$, subsets of the same $\widetilde{T}$ orbit, it follows that

$$d_T(E, F) \equiv \inf\{\|v\| : E \cap \widetilde{T}^v F \neq \emptyset\}.$$

4.5. **Tilings by action polygons.** Suppose that $C \subset X$ is a clopen set. For each $x \in X$ we may consider the set $\mathcal{V}_x = \{v \in \mathbb{Z}^2 : T^v x \in C\} \subset \mathbb{R}^2$, or more specifically, $\{x\} \times \mathcal{V}_x \subset X \times \mathbb{R}^2$. Now, if $y = T^u x$ for some $u \in \mathbb{Z}^2$, then $\Pi_T(\{x\} \times \mathcal{V}_x) = \Pi_T(\{y\} \times \mathcal{V}_y)$. We will show (in Lemma 20) that C can be chosen so that $\mathcal{V}_x$ is m-regular for any $x \in X$. Thus for each $x \in X$ we may create a Delaunay tiling $\mathfrak{D}(\mathcal{V}_x)$ of

$\mathbb{R}^2$ based on the vertex set $\mathcal{V}_x$ and the geometry of each such tiling satisfies properties from Facts 15 and 16 and Remark 18. By constructing tilings in this way, the map $x \mapsto \mathfrak{D}(\mathcal{V}_x)$ commutes with the actions (T on X and translation on the tiling space) and is continuous. Considering $\mathfrak{D}(\mathcal{V}_x)$ as a tiling of $\{x\} \times \mathbb{R}^2$, we may project the set of all tilings of the form $\mathfrak{D}(\mathcal{V}_x)$ to X_T via Π_T. The result is what we refer to in the next subsection as a *tiling of X_T by action polygons*. In the remainder of this section, we formalize this idea, then prove that we can produce such an object satisfying the desired geometric properties, culminating in the Tiling Lemma (Lemma 22)

Definition 19. Suppose we have a Cantor subset $\mathbf{V}_T \subset X_T$, a clopen partition $\{\mathbf{V}_i\}_{i=1}^I$ of $\mathbf{V}_T$ and set of polygons $\{P_i\}_{i=1}^I$ such that

1. the collection of sets $\mathbf{P}_i = \{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\}$ for $i \in 1, .., I$ are closed and cover X_T,
2. the sets $\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in \mathit{Interior}(P_i)\}$ are open and pairwise disjoint in X_T,
3. for $i \neq j$, $\mathbf{P}_i \cap \mathbf{P}_j$ is a finite union of sets of the form $\{\widetilde{T}^v \mathbf{U}_k : v \in e_k\}$ where $\mathbf{U}_k$ is a Cantor set and each e_k is a translate of a *face* (an edge or vertex) from both P_i and P_j.

We call the triple $\mathbf{P}_T = (\mathbf{V}_T, \{\mathbf{V}_i\}, \{P_i\})$ a *tiling of X_T by action polygons.*

There are two immediate examples of such a triple. 1) $\mathbf{V}_T = \Pi_T(X, 0)$, $\{\mathbf{V}_i\}$ consists of the single set $\mathbf{V}_1 = \mathbf{V}_T$ and $\{P_i\}$ consists of the single tile $P_1 = [0,1] \times [0,1]$. With these choices, the definition of a tiling by action polygons only requires that $\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_1, r \in \mathit{Interior}(P_1)\}$ be open, which it is.

A little more substantial is 2) where we let $\{\mathbf{V}_i\}_{i=1}^I$ be any clopen partition of $\Pi_T(X, 0)$ and $\{P_i\}$ consists of the single tile $[0,1] \times [0,1]$.

4.6. **Voronoi and Delaunay tilings by action polygons.** Our goal is to construct tilings of X_T by action polygons derived from Voronoi and Delaunay tilings of the acting space $\mathbb{R}^2$. The following is a variation of a well-known lemma [K1, LM, L1] which has probably been proven elsewhere.

Lemma 20. *Let (X, T) be a $\mathbb{Z}^2$ minimal Cantor system. For any $m \geq 1$ there exists a clopen set $C \subset X$ such that for each $x \in X$ the set $\mathcal{V}_x = \{v \in \mathbb{Z}^2 : T^v x \in C\}$ is an m-regular subset of $\mathbb{Z}^2$.*

Proof. Our goal is to construct a clopen set $C \subset X$ with the following properties.

1. for any $x \in X$, $\{T^v x : \|v\| \leq m\} \cap C \neq \emptyset$,
2. for any $x \in C$, $\{T^v x : 0 < \|v\| \leq m\} \cap C = \emptyset$.

Since $T^v x = x$ implies $v = 0$ and the space X is compact, we can let

$$\delta = \min\{d(x, T^v x) : \|v\| \leq m\} > 0.$$

For each $x \in X$, let E_x be a clopen set containing x with diameter less than δ. Since the space X is compact there is a finite subcover $E_1, E_2, \ldots, E_n$ of such clopen sets.

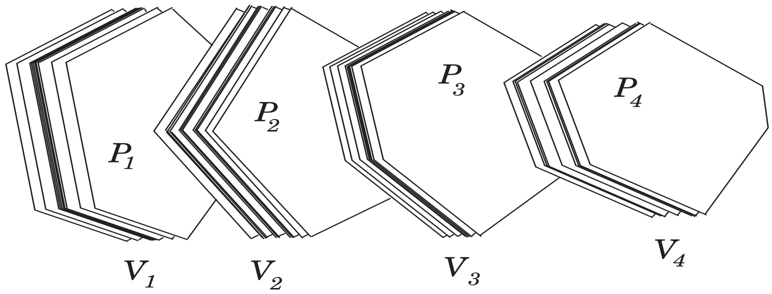


FIGURE 3. The structure of X_T, a flow in a finite set of polygons.

We may assume that the sets E_j are pairwise disjoint, otherwise substitute E_j with $E_j \setminus \bigcup_{i<j} E_i$. Thus we have a finite collection of pairwise disjoint clopen sets $\{E_j\}$ such that $X = \bigcup E_j$ and $diam(E_j) < \delta$ for all j. Let $F_1 = E_1$ and for $j > 1$, let $F_j = F_{j-1} \cup \left(E_j \setminus \bigcup_{\|v\| \leq m} T^v F_{j-1}\right)$. Let $C = F_n$.

We now prove property (1). Let $x \in X$. Then $x \in E_j$ for some $1 \leq j \leq n$. Either $x \in F_j$, in which case $x \in C$, or $x \in \bigcup_{\|v\| \leq m} T^v F_{j-1}$. In the first case we are done, so assume the latter case. Then there is a $x' \in F_{j-1}$ and a vector v with $\|v\| \leq m$ such that $x = T^v x'$, or $T^{-v} x = x' \in C$. This implies $\{T^v x : \|v\| \leq m\} \cap C \neq \emptyset$.

To prove property (2) assume $x \in C$. There is a $j > 0$ such that $x \notin F_k$ for $k < j$ and $x \in F_k$ for all $k \geq j$. Let $x' = T^v x$ for some vector v with $\|v\| \leq m$. Then $x' \notin F_k$ for $k < j$ since that would imply $x' \in F_{j-1}$ which would imply that $x \notin F_j$. Further, $x' \notin F_j$ since this would imply $x' \in E_j$, but $diam(E_j) < \delta = \min\{d(x, T^v x) : \|v\| \leq m\}$, a contradiction. Finally, if $k > j$ then $x' \notin F_k$ since $x' \in \bigcup_{\|v\| \leq m} T^v F_{k-1}$ for all $k > j$. Therefore, $\{T^v x : 0 < \|v\| \leq m\} \cap C = \emptyset$. □

For $r > 0$, define $\mathcal{V}_x(r) = \{v \in \mathcal{V}_x : \|v\| < r\}$. Note that for any $r > 0$, the map $x \mapsto \mathcal{V}_x(r)$ is locally constant since $\mathcal{V}_x(r) = \{v : \|v\| < r \text{ and } T^v x \in C\}$ and C is clopen. This is the sense in which the map $x \mapsto \mathcal{V}_x$ is continuous. Also note that the map $x \mapsto \mathcal{V}_x$ commutes with the actions since $\mathcal{V}_{T^v x} = \mathcal{V}_x - v$.

The following result is related to the Tiling Lemma (Lemma 22) and may be of interest in its own right as a description of the structure of X_T. (See Figure 3.) The proof of Lemma 21 foreshadows that of the Tiling Lemma.

Lemma 21. *(A Voronoi action tiling of X_T) Let (X,T) be a $\mathbb{Z}^2$ minimal Cantor system. Given $m \geq 1$, there exist a Cantor set $\mathbf{V}_T \subset X_T$, a clopen partition $\{\mathbf{V}_i\}_{i=1}^I$ of $\mathbf{V}_T$, and a finite set of convex polygons $P_i \subset \mathbb{R}^2$, $i = 1, .., I$ such that $(\mathbf{V}_T, \{\mathbf{V}_i\}, \{P_i\})$ is a finite tiling of X_T by convex action polygons. Moreover,*

(1) *for $\widetilde{x} \in \mathbf{V}_T$ the sets $\{r \in \mathbb{R}^2 : \widetilde{T}^r \widetilde{x} \in \mathbf{V}_T\}$ are subsets of $\mathbb{Z}^2$ and m-regular,*
(2) *for each $\widetilde{x} \in \mathbf{V}_i$, $\{\widetilde{T}^r \widetilde{x} : r \in P_i\} = \{\widetilde{z} \in X_T : d_T(\widetilde{x}, \widetilde{z}) \leq d_T(\widetilde{x}, \mathbf{V}_T)\}$,*
(3) *$\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in Interior(P_i)\} = Interior(\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\})$.*

Proof. Given $m > 0$, let C and $\mathcal{V}_x$ be from Lemma 20. First we define $(\mathbf{V}_T, \{\mathbf{V}_i\}, \{P_i\})$ and prove it is a finite tiling of X_T by convex action polygons.

Define $\mathbf{V}_T = \{\Pi_T(x, \mathcal{V}_x) : x \in X\}$. Note that $\mathbf{V}_T$ is also equal to the set $\{\Pi_T(x, 0) : x \in C\}$. Because Π_T is injective and continuous on $C \times \{0\}$, $\mathbf{V}_T$ is a Cantor set. Let $\{P_i\}_{i=1}^I = \{V_v - v : v \in \mathcal{V}_x, x \in X\}$. For $x \in C$, $0 \in \mathcal{V}_x$; let $V_0(\mathcal{V}_x)$ be the tile in $\mathfrak{V}(\mathcal{V}_x)$ which contains the origin, and let $C_i = \{x \in C : V_0(\mathcal{V}_x) = P_i\}$. That is, C_i is the collection of $x \in C$ for which the Voronoi tile corresponding to 0 is P_i.

Letting $\mathbf{V}_i = \Pi_T(C_i \times \{0\})$ and setting $\mathbf{P}_i = \{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\}$ we have

(1) the sets $\mathbf{P}_i$ for $i \in 1, .., I$ are closed and cover X_T,
(2) the sets $\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in Interior(P_i)\}$ are open and pairwise disjoint,

because the same properties hold above, in the preimage of Π_T. This gives us the first two conditions for a tiling by action polygons.

That the third condition for a tiling by action polygons holds is a consequence of the fact that the Voronoi polygons (in $\mathbb{R}^2$ derived from each $\mathcal{V}_x$) have pairwise intersections that are faces (when non-empty). Suppose $\widetilde{x} \in \mathbf{P}_i \cap \mathbf{P}_j$. We will show that there is a Cantor set $\mathbf{U}_k$ and a set e_k (which is a translate of a face from both P_i and P_j) such that $\widetilde{x} \in \{\widetilde{T}^r \widetilde{c} : \widetilde{c} \in \mathbf{U}_k, r \in e_k\} \subset \mathbf{P}_i \cap \mathbf{P}_j$ and that finitely many such sets cover $\mathbf{P}_i \cap \mathbf{P}_j$.

Because $\widetilde{x} \in \mathbf{P}_i \cap \mathbf{P}_j$, there exists $\widetilde{x}_i \in \mathbf{V}_i$, $\widetilde{x}_j \in \mathbf{V}_j$ and $r_i \in P_i$, $r_j \in P_j$ such that $\widetilde{T}^{r_i} \widetilde{x}_i = \widetilde{x} = \widetilde{T}^{r_j} \widetilde{x}_j$. There exist $a \in C_i$ and $b \in C_j$ such that $\widetilde{x}_i = \Pi_T(a, 0)$ and $\widetilde{x}_j = \Pi_T(b, 0)$. It follows that $\Pi_T(a, r_i) = \Pi_T(b, r_j)$. Because Π_T maps (a, r_i) and (b, r_j) to the same point, there exists $u, v \in \mathbb{Z}^2$ such that $T^u a = T^v b$ and $r_i - u = r_j - v$. So, $r_i - r_j = u - v \in \mathbb{Z}^2$ and $b = T^{r_i - r_j} a$, which further implies that $r_i - r_j \in \mathcal{V}_a$. Letting $r = r_i - r_j$ we set $C_k = C_i \cap T^{-r} C_j$ and note that C_k is non-empty and clopen.

The element $r_i \in P_i \cap (r + P_j)$ because $r_j \in P_j$ and $r = r_i - r_j$. By definition (of C_i), P_i is the Vononoi tile in $\mathfrak{V}(\mathcal{V}_a)$ that contains the origin and likewise P_j is the tile in $\mathfrak{V}(\mathcal{V}_b)$ that contains the origin. Because $b = T^r a$, $\mathfrak{V}(\mathcal{V}_a) = \mathfrak{V}(\mathcal{V}_b) + r$. So the tile in $\mathfrak{V}(\mathcal{V}_b)$ containing the origin, P_j, after shifting by r, is the tile in $\mathfrak{V}(\mathcal{V}_a)$ which contains r. So, because $r \in \mathcal{V}_a$, we may write for the tile V_r in $\mathfrak{V}(\mathcal{V}_a)$ which corresponds to r that $V_r = r + P_j$. Thus we have shown that $V_0, V_r \in \mathfrak{V}(\mathcal{V}_a)$ are such that $V_0 \cap V_r \neq \emptyset$, because $V_0 = P_i$, $V_r = P_j + r$ and $P_i \cap (P_j + r) \neq \emptyset$.

Because P_i and $r + P_j$ are intersecting Voronoi tiles, they intersect in a face, which we label e_k. Notice e_k is a face in P_i and it is a translate of a face in P_j ($e_k - r$ is a face in P_j). So $r_i \in e_k$, $r_j \in e_k - r$ and $\widetilde{x} \in \{\widetilde{T}^t \Pi_T(a, 0) : t \in e_k\} = \{\widetilde{T}^s \Pi_T(b, 0) : s \in e_k - r\}$ which is a subset of $\{\widetilde{T}^t \Pi_T(c, 0) : c \in C_k, t \in e_k\} = \{\widetilde{T}^s \Pi_T(d, 0) : d \in T^r C_k, s \in e_k - r\}$, a subset of $\mathbf{P}_i \cap \mathbf{P}_j$. Letting $\mathbf{U}_k = \Pi_T(C_k \times \{0\})$ we have that $\mathbf{U}_k$ is a Cantor set in X_T and that $\{\widetilde{T}^t \Pi_T(c, 0) : c \in C_k, t \in e_k\} = \{\widetilde{T}^t \mathbf{U}_k : t \in e_k\}$.

The number of such vectors $r \in \mathbb{Z}^2$ and such sets e such that $e = P_i \cap (P_j + r) \neq \emptyset$ is finite, due to their origins as Vononoi tilies in $\mathbb{R}^2$ with respect to m-regular subsets of $\mathbb{Z}^2$. Thus there exists a finite number of sets $\mathbf{U}_k = \Pi_T((C_i \cap T^{-r}C_j) \times \{0\})$ and e_k such that $\{\widetilde{T}^r \mathbf{U}_k : r \in e_k\}$ cover $\mathbf{P}_i \cap \mathbf{P}_j$, giving us the third condition required to have a tiling by actions polygons.

Condition (1) of *this* lemma follows because $\{r \in \mathbb{R}^2 : \widetilde{T}^r \widetilde{x} \in \mathbf{V}_T\} = \{v \in \mathbb{R}^2 : \widetilde{T}^v \Pi_T(x,0) \in \Pi_T(x, \mathcal{V}_x)\} = \mathcal{V}_x$ where $\Pi_T(x,0) = \widetilde{x}$. Condition (2) holds because the same property holds in the Π_T preimage. In order for the third condition to hold we need to make sure elements in $\mathbf{P}_i$ of the form $\widetilde{T}^r \Pi_T(c,0)$ for $c \in C_i$ and $r \in \partial P_i$ are not elements of the interior of $\mathbf{P}_i$. This will be the case if for each $c \in C_i$ and $u, v \in \mathcal{V}_c$, we have that $(u + V_u) \cap (v + V_v) = \emptyset$ where V_u and V_v are the tiles in $\mathfrak{V}(\mathcal{V}_c)$ corresponding to u and v. This will hold uniformly for $u, v, c \in C_i$ and i by refining the sets C_i to have small diameter (with respect to the metric in X), because the diameters of the P_i (with respect to the metric in $\mathbb{R}^2$), and thus the V_u and V_v, are all bounded by $2m + 2$, and because the action is free. □

The final result of this section, the Tiling Lemma, is a result analogous to Lemma 21. The Tiling Lemma is based on Delaunay tilings instead of Voronoi tilings used in Lemma 21. The proof of the Tiling Lemma uses the same sorts of ideas as Lemma 21 but is slightly more difficult to explain. In particular, we wish to refer to geometric properties of the graph corresponding to the boundaries of tilings of X_T.

By a *graph embedded in* X_T we mean a graph $G_T = (\mathbf{V}_T, \mathbf{E}_T)$ with vertex set $\mathbf{V}_T$ (of arbitrary cardinality) and edge set $\mathbf{E}_T$ with the following two properties.

(1) Each edge $\widetilde{e} \in \mathbf{E}_T$ is a path (*i.e.* a continuous, injective image of $[0,1]$ in X_T) and for distinct $\widetilde{e}, \widetilde{f} \in \mathbf{E}_T$, $\widetilde{e} \cap \widetilde{f}$ is either empty or a single point which is an endpoint of both $\widetilde{e}$ and $\widetilde{f}$.

(2) $\mathbf{V}_T$ is the set of endpoints of the edges of $\mathbf{E}_T$.

Because the path-connected components of X_T are the $\widetilde{T}$ orbits, each edge is a subset of a single $\widetilde{T}$ orbit, and so, of course, each connected component of the graph is also a subset of a single orbit in $\widetilde{T}$. Given $\widetilde{v} \in \mathbf{V}_T$, we define $star(\widetilde{v}) = \cup\{\widetilde{e} : \widetilde{e} \in \mathbf{E}_T$ and $\widetilde{v} \in \widetilde{e}\}$. That is, $star(\widetilde{v})$ is the union of all the edges that contain $\widetilde{v}$ as an endpoint.

Each $\widetilde{e} \in \mathbf{E}_T$ is a line segment, so if $\widetilde{e} \cap \widetilde{f} \neq \emptyset$ then we may take the meaning of the angle $\angle(\widetilde{e}, \widetilde{f})$ between $\widetilde{e}$ and $\widetilde{f}$ from the angle between the lifts of $\widetilde{e}$ and $\widetilde{f}$ in a common $\{x\} \times \mathbb{R}^2$.

Let $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \{\widehat{\mathbf{V}}_i\}, \{D_i\})$ be a tiling of X_T by action polygons (the choice of symbols may seem unnatural, but these are the symbols that will be used when this definition is applied.) By the *boundary* $\partial \mathbf{D}_T$ of $\mathbf{D}_T$ we mean the following set

$$\partial \mathbf{D}_T = \cup\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \widehat{\mathbf{V}}_i, r \in \partial D_i\}.$$

We say a graph $G_T = (\mathbf{V}_T, \mathbf{E}_T)$ embedded in X_T *tiles* ∂D_T if $\cup\{\widetilde{e} : \widetilde{e} \in \mathbf{E}_T\} = \partial \mathbf{D}_T$. We use the word *tiles* here because we are viewing the endpoints of each edge as the boundary of that edge and we are viewing the interior of the edge as the edge minus its endpoints. Distinct edges have disjoint interiors because distinct edges only intersect at their endpoints, if at all. So, the objects that cover $\partial \mathbf{D}_T$ are closed and have disjoint interiors; they tile $\partial \mathbf{D}_T$. Let us note that the set $\mathbf{V}_T$ in Lemma 22 is the same set $\mathbf{V}_T$ found in Lemma 21.

Lemma 22 (The Tiling Lemma)**.** *There exists a $\delta > 0$ such that the following holds. Let $K > 0$ be given. For all sufficiently large m, there exists a tiling of X_T by action polygons $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \{\widehat{\mathbf{V}}_i\}, \{D_i\})$ and a graph $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ embedded in X_T that tiles $\partial \mathbf{D}_T$ in which $\mathbf{V}_T \subset \Pi_T(X \times \mathbb{Z}^2)$ with the following properties.*

1. *$\forall \widetilde{x} \in \mathbf{V}_T$ the set $\mathcal{V}_{\widetilde{x}} = \{v \in \mathbb{R}^2 : \widetilde{T}^v \widetilde{x} \in \mathbf{V}_T\} \subset \mathbb{Z}^2$ and is m-regular in $\mathbb{Z}^2$,*
2. *there exist a finite clopen partition $\{\mathbf{S}_i\}$ of $\mathbf{V}_T$ and a finite collection $\{s_i\}$ where each s_i is a finite set of line segments each of which contains the origin such that for each $\widetilde{x} \in \mathbf{S}_i$, $star(\widetilde{x}) = \{\widetilde{T}^r \widetilde{x} : r \in s_i\}$,*
3. *for each edge $e = [\widetilde{x}, \widetilde{y}] \in \mathbf{E}_T$, $m \leq d_T(\widetilde{x}, \widetilde{y}) \leq 2(m+1)$,*
4. *for each pair of distinct edges $e, f \in \mathbf{E}_T$, $e \cap f \neq \emptyset$ implies $\angle(e, f) > \delta$,*
5. *for any pair of Delaunay edges $l, l' \in \mathbf{E}_T$, either $l \cap l' \neq \emptyset$ or $d_T(l, l') > 2K$.*

Proof. Let $\delta > 0$ be from Fact 15. For $K > 0$, let m_K be from Fact 16. Assume that $m > m_K$. As before, let C and $\mathcal{V}_x$ be from Lemma 20 with this value of m. Add a subscript x to the notation of Section 4.3 to get $\widehat{\mathcal{V}}_x = \{p \in \mathbb{R}^2 : |\mathcal{V}_x(p)| \geq 3\}$. Set $\mathbf{V}_T = \{\Pi_T(x, \mathcal{V}_x) : x \in X\}$ and $\widehat{\mathbf{V}}_T = \{\Pi_T(x, \widehat{\mathcal{V}}_x) : x \in X\}$. To show that $\widehat{\mathbf{V}}_T$ is a Cantor set, let $I^2 = [0,1) \times [0,1)$ and let $Q = \{q : q \in \widehat{\mathcal{V}}_x \cap I^2$ for some $x \in X\}$. The set Q is finite because $Q \subset \{q : q \in \widehat{\mathcal{V}} \cap I^2$ for some $\mathcal{V} \in \mathfrak{M}_m\}$, which is finite (recall the discussion in Section 4.3). Because $\widehat{\mathbf{V}}_T$ is a closed subset of $\{\Pi_T(x, Q) : x \in X\}$, $\widehat{\mathbf{V}}_T$ is zero-dimensional. In order to show $\widehat{\mathbf{V}}_T$ is a Cantor set we need to show it has no isolated points, and this follows from the continuity of the maps $x \mapsto \mathcal{V}_x$ and $\mathcal{V}_x \mapsto \widehat{\mathbf{V}}_T$.

For $x \in X$, we define $\mathfrak{D}(\mathcal{V}_x) = \{\mathcal{H}(\mathcal{V}_x(p)) : p \in \widehat{\mathcal{V}}_x\}$ and we write $\{D_i\}_{i=1}^J = \{\mathcal{H}(\mathcal{V}_x(p)) - p : p \in \widehat{\mathcal{V}}_x, x \in X\}$. (The latter set is finite since all sets $\mathcal{V}_x$ with $x \in X$ are m-regular.) Thus for each $x \in X$, $\mathfrak{D}(\mathcal{V}_x)$ is a finite tiling of $\mathbb{R}^2$ with convex polygons from the list $\{D_i\}_{i=1}^J$. Let $\widehat{\mathbf{V}}_i = \left\{\Pi_T(x,p) : p \in \widehat{\mathcal{V}}_x, \mathcal{H}(\mathcal{V}_x(p)) - p = D_i\right\}$. Each such set $\widehat{\mathbf{V}}_i$ is a Cantor set because the map $x \mapsto \mathcal{V}_x$ is continuous and the m-regularity of $\mathcal{V}_x$ implies that $d(p, v) < m + 1$ where $v \in \mathcal{V}_x(p)$. The remainder of the proof that $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \{\widehat{\mathbf{V}}_i\}, \{D_i\})$ is a tiling of X_T by action polygons follows from arguments similar to those in Lemma 21. Property 1 holds for exactly the same reason it does in Lemma 21; because $\mathbf{V}_t$ is exactly the same set here as there.

The boundary of $\mathfrak{D}(\mathcal{V}_x)$ is the set $\partial\mathfrak{D}(\mathcal{V}_x) = \cup_{p \in \widehat{\mathcal{V}}_x} \partial\mathcal{H}(\mathcal{V}_x(p))$, and it is tiled by the 1-dimensional faces of all the $\mathcal{H}(\mathcal{V}_x(p))$. Namely, $\mathcal{E}_x = \{p + e : e$ is a 1-dimensional face in $\mathcal{H}(\mathcal{V}_x(p))$, $p \in \widehat{\mathcal{V}}_x\}$ is the set of line segments that tiles $\partial\mathfrak{D}(\mathcal{V}_x)$. Moreover, the set of endpoints of the line segments in $\mathcal{E}_x$ is $\mathcal{V}_x$. So, there is a graph with vertices $\mathcal{V}_x$ and edges $\mathcal{E}_x$ (which we call $G_x = (\mathcal{V}_x, \mathcal{E}_x)$). This is the graph of the boundary

$\partial\mathfrak{D}(\mathcal{V}_x)$. The elements of $\mathcal{E}_x$ are called Delaunay edges, and the elements of $\mathcal{V}_x$ are called the Delaunay vertices. Set $\mathbf{E}_T = \{\Pi_T(x, \mathcal{E}_x) : x \in X\}$. We have already explained (in the proof of Lemma 21) that $\mathbf{V}_T$ is a Cantor set, and it follows along the same lines as that proof that $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ is a graph embedded in X_T that tiles $\partial\mathbf{D}_T$.

From Section 4.3 one can see that the collection

$$S = \{star(v) - v : v \in \mathcal{V}, \text{ and } \mathcal{V} \in \mathfrak{M}_m\}$$

is finite. (Each element $s \in S$ is a finite union of the form $s = \cup\{[0, v_j] : v_j \in \mathbb{R}^2\}$.) Because S is finite, we write $S = \{s_i\}$.

Let $\mathcal{V}_x^i = \{v \in \mathcal{V}_x : star(v) = s_i + v\}$. Because $v \in \mathcal{V}_x$ if and only if $T^v x \in C$ and $z \in C$ if and only if $0 \in \mathcal{V}_z$, one can check that for $U_i = \{x \in C : star(0) = s_i\}$ we have

$$\mathbf{S}_i = \{\Pi_T(x, \mathcal{V}_x^i) : x \in X\} = \{\Pi_T(x, 0) : x \in U_i\}.$$

Because of the continuity of the map $x \mapsto \mathcal{V}_x$ and because $\mathcal{E}_x$ is locally determined by $\mathcal{V}_x$, the set U_i is clopen, and as a clopen subset of a Cantor set, U_i is a Cantor set. Because Π_T is injective on $X \times \{0\}$, $\mathbf{S}_i$ is also a Cantor set. Because $\cup_i \mathcal{V}_x^i = \mathcal{V}_x$, the sets $\mathbf{S}_i$ cover $\mathbf{V}_T$. Thus the $\mathbf{S}_i$ are a finite clopen partition of $\mathbf{V}_T$.

By construction each s_i is a finite set of line segments that each contain the origin. For each $\widetilde{x} \in \mathbf{S}_i$ there a unique $x \in U_i$ such that $\widetilde{x} = \Pi_T(x, 0)$. The set of edges $\widetilde{e} \in \mathbf{E}_T$ which intersect $\widetilde{x}$ are the Π_T images of a set of edges in $(x, \mathcal{E}_x)$, specifically the set of edges $(x, e) \in (x, \mathcal{E}_x)$ that intersect the origin. The union of those edges, by dint of the fact that $x \in U_i$, is s_i. Thus $star(\widetilde{x}) = \{\Pi_T(x, r) : r \in s_i\} = \{\widetilde{T}^r \widetilde{x} : r \in s_i\}$, which is the desired equality. This gives property (2)

The geometric properties (3)-(5) follows from Facts 15 and 16 and Remark 18 for Delaunay tilings corresponding to m-regular subsets of $\mathbb{Z}^2$. □

Remark 23. With the exception of property (5) in Lemma 22 above, the previous two lemmas (Lemma 21, 22) can be proven for minimal $\mathbb{Z}^d$ actions and tilings in $\mathbb{R}^d$ for $d > 2$. However, this geometric property (5) is critical for the construction in the next section. The reason is the following. The properties of the bounded orbit equivalence, and the related map g are such that points in X_T which are far apart with respect to d_T are mapped to points in Y_S which are far with respect to d_S (recall Corollary 14.1″). Thus, we have a dichotomy in the tilings: either two vertices/edges/tiles in X_T intersect or are far apart. Furthermore, these properties are preserved by the map g. Therefore to "repair" g to be injective, we can simply work locally, considering one vertex/edge/tile at a time.

In a higher dimensional setting, we do not know how to arrange this situation. For example, we would like to avoid situations like the following. Suppose there is a tiling of $\mathbb{R}^3$ which contains the tetrahedron with vertices $u = (n, 0, 0)$, $v = (-n, 0, 0)$, $w = (0, n, 1)$, $z = (0, -n, 1)$. The vertices are regular in the sense of their distances apart, but the edges $[u, v]$ and $[w, z]$ which are non-intersecting are nevertheless close together. Therefore we cannot guarantee anything about their images under the linear extension of a bounded orbit equivalence, and it becomes unclear when working in the image space how one should repair the lack of injectivity. In Section 7 is an example of an m-regular subset of $\mathbb{Z}^3$ in which there are Delaunay edges that exhibit the above property.

5. Graph Isomorphism Lemma

We now have a tiling of X_T by action polygons which satisfies some important geometric properties. The next step is to consider the image of the tilings, or more specifically, the boundaries of the tilings under the map g constructed in Section 3.1. To this end, let us say that a map $\phi : \partial\mathbf{D}_T \to Y_T$ is *orbit injective* if for any $\widetilde{x}, \widetilde{y} \in \partial\mathbf{D}_T$ and $v \in \mathbb{R}^2$, $\phi(\widetilde{x}) = \widetilde{S}^v\phi(\widetilde{y})$, then $\widetilde{x} = \widetilde{T}^u\widetilde{y}$ for some $u \in \mathbb{R}^2$, and conversely, $\widetilde{x} = \widetilde{T}^u\widetilde{y}$ implies $\phi(\widetilde{x}) = \widetilde{S}^v\phi(\widetilde{y})$ for some $v \in \mathbb{R}^2$.

The main goal of this section is to prove the Graph Isomorphism Lemma.

Lemma 24 (Graph Isomorphism Lemma)**.** *Given a continuous bounded orbit injection $h : X \to Y$, there exist a tiling of X_T by action polygons $\mathbf{D}_T$, a graph $G(\mathbf{D}_T)$, and a continuous, injective, orbit injective map $\phi : \partial\mathbf{D}_T \to Y_S$ such that*

1. *$\mathbf{D}_T$ and $G(\mathbf{D}_T)$ satisfy the conditions of the Tiling Lemma 22,*
2. *ϕ is a graph isomorphism on $(\mathbf{V}_T, \mathbf{E}_T)$,*
3. *$\phi(\mathbf{V}_T) \subset \Pi_S(Y \times \mathbb{Z}^2)$,*
4. *there is a finite clopen partition $\{\mathbf{Q}_k\}$ of $\mathbf{V}_T$ which refines the partition $\{\mathbf{S}_i\}$ from Lemma 22 such that for each $\mathbf{Q}_k \subset \mathbf{S}_i$ there is a map $\widehat{\alpha}_k : s_i \to \mathbb{R}^2$ such that for all $\widetilde{y} \in \mathbf{Q}_k$ and $v \in s_i$*
$$\phi(\widetilde{T}^v\widetilde{y}) = \widetilde{S}^{\widehat{\alpha}_k(v)}\phi(\widetilde{y}).$$

Let ∂g be the restriction of g (from Section 3.1) to $\partial\mathbf{D}_T$ viewed as a map from $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ to Y_S, where $\mathbf{D}_T$ and $G(\mathbf{D}_T)$ are from the Tiling Lemma. It (∂g) satisfies all the properties desired for ϕ in Lemma 24 except Part (2). The map ∂g is not a graph homomorphism, much less a graph isomorphism. Our goal in this section is to perturb ∂g so that it is injective on $\partial\mathbf{D}_T$. We will call the perturbation ϕ and it will be the graph isomorphism sought in the Graph Isomorphism Lemma 24.

5.1. **Choosing the size of m.** In order to have all the desired machinery up and running for the perturbation, we need an appropriate value for m. We now describe how to find m (see Figure 4).

Let $\delta > 0$ be as in the Tiling Lemma (recall that δ was a positive lower bound on the angle between two intersecting edges in a Delaunay tiling derived from an m-regular vertex set). Corollary 14.1$''$ with $M = 0$ gives us a number N_0 such that if $d_T(\widetilde{x}, \widetilde{y}) > N_0$ then $g(\widetilde{x}) \neq g(\widetilde{y})$. Let R_1 and R_2 be (any) two rays emanating from the origin such that the angle between them is bounded below by δ. Define $r > N_0$

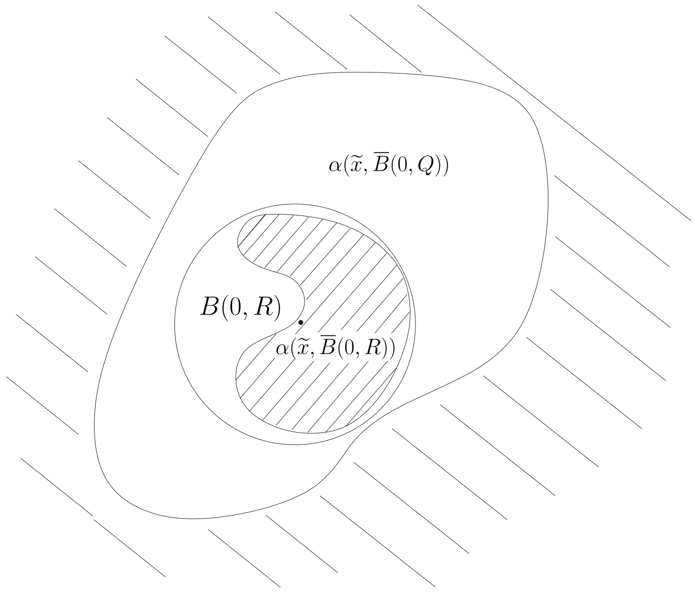


FIGURE 4. Choosing m

to be such that if $u_1 \in R_1$ and $u_2 \in R_2$ are any two points with $\|u_i\| \geq r$, then $\|u_1 - u_2\| > N_0$.

Because of Proposition 12″, $\alpha(\widetilde{x}, \overline{B}(0,r))$ is uniformly bounded for $\widetilde{x} \in X_T$, so there exists an $R > 0$ such that $\alpha(\widetilde{x}, \overline{B}(0,r)) \subset B(0,R)$ for all $\widetilde{x} \in X_T$. By Corollary 14.1″ there exists $Q = N_R \geq r$ such that (1) for all $\widetilde{x} \in X_T$ and all v with $\|v\| > Q$, $\|\alpha(\widetilde{x}, v)\| > R$ (with the by-product $Q = N_R \geq r > N_0$), and (2) for all $\widetilde{x} \in X_T$ and all v with $\|v\| > 2Q$, $\|\alpha(\widetilde{x}, v)\| > 2R$.

Now apply the Tiling Lemma with $K = Q$ to obtain a value of m sufficiently large such that there is a tiling $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \{\widehat{\mathbf{V}}_i\}, \{D_i\})$ of X_T by action polygons with the associated graph $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ and partition $\{\mathbf{S}_i\}_{i=1}^K$ of $\mathbf{V}_T$ satisfying conditions (1)-(5). Note that $m > 2Q \geq 2r$. All such elements, including the constants m, Q, R and r are fixed from this point forward.

5.2. **Preliminary results dependent on the choice of** m**.** Let $e \in \mathbf{E}_T$ be an edge with the endpoints $\widetilde{x}$ and $\widetilde{y}$, and define

$$e(\widetilde{x}, r) = \{\widetilde{z} \in e : d_T(\widetilde{x}, \widetilde{z}) \geq r\}.$$

Since $d_T(\widetilde{x}, \widetilde{y}) \geq m > 2r$, the set $e(\widetilde{x}, r)$ is nonempty.

The following two propositions are used in the proof of Lemma 30. The propositions follow from our choice of m.

Proposition 25. *For any distinct* $e, e' \in \mathbf{E}_T$*, if* $e \cap e' = \widetilde{c}$*, then* $\partial g(e(\widetilde{c}, r)) \cap \partial g(e'(\widetilde{c}, r)) = \emptyset$.

Proof. This follows from the existence of δ and our choice of r. Let $\widetilde{x} \in e(\widetilde{c}, r)$ and $\widetilde{x}' \in e'(\widetilde{c}, r)$. Then there exist $v, v' \in \mathbb{R}^2$ such that $\|v\|, \|v'\| \geq r$ and such that $\widetilde{x} = \widetilde{T}^v\widetilde{c}$ and $\widetilde{x}' = \widetilde{T}^{v'}\widetilde{c}$. By our choice of r, $\|v - v'\| > N_0$, which means $\partial g(\widetilde{x}) = \partial g(\widetilde{T}^v\widetilde{c}) = \widetilde{S}^{\alpha(\widetilde{c},v)}\partial g(\widetilde{c}) \neq \widetilde{S}^{\alpha(\widetilde{c},v')}\partial g(\widetilde{c}) = \partial g(\widetilde{T}^{v'}\widetilde{c}) = \partial g(\widetilde{x}')$, where the inequality in the middle is provided by Corollary 14.1″. Thus, the lemma follows. ☐

Proposition 26. *For any $\widetilde{x} \in X_T$ and $u, v \in \mathbb{R}^2$,*

$$\|u - v\| > 2Q \text{ implies } \|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > 2R.$$

In particular, $\|u - v\| \geq m$ implies $\|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > 2R$.

Proof. By assumption $v - u \notin \overline{B}(0, 2Q)$, and by our choice of Q, $\alpha(\widetilde{T}^u\widetilde{x}, v - u) \notin \overline{B}(0, 2R)$. By the cocycle equation $\alpha(\widetilde{T}^u\widetilde{x}, v - u) = \alpha(\widetilde{x}, v) - \alpha(\widetilde{x}, u)$, so $\|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > 2R$. ☐

5.3. **A Planar Graph Isomorphism: the Perturbation of ∂g.** We have established that $\partial g : \partial\mathbf{D}_T \to Y_S$ is a continuous, orbit injective map. By its construction $\mathbf{V}_T \subset \Pi_S(X \times \mathbb{Z}^2)$ and $\partial g(\mathbf{V}_T) \subset \Pi_S(Y \times \mathbb{Z}^2)$. For each $\widetilde{x} \in \mathbf{S}_i \subset \mathbf{V}_T$ the map $\alpha(\widetilde{x}, \cdot)$ maps s_i to $\mathbb{R}^2$. Because the line segments in s_i are uniformly bounded in length, and because α is an extension of the continuous function β that is locally constant in the first component, the set $\mathbf{S}_i$ is the union of finitely many clopen sets $\{\mathbf{Q}_{(i,j)}\}$ such that $\alpha(\widetilde{x}, s) = \alpha(\widetilde{y}, s)$ for $\widetilde{x}, \widetilde{y} \in \mathbf{Q}_{(i,j)}$ and $s \in s_i$. For simplicity, reindex the sets $\mathbf{Q}_{(i,j)}$ as $\mathbf{Q}_k$ and the corresponding sets s_i as s_k. Set α_k equal to the restriction of $\alpha(\widetilde{x}, \cdot)$ to s_k for any $\widetilde{x} \in \mathbf{Q}_k$. The maps $\widehat{\alpha}$ we construct in the Graph Isomorphism Theorem (4) will be perturbations of these maps.

Let $G = (\mathcal{V}, \mathcal{E})$ be a planar graph and let $|G| = \cup\{e : e \in \mathcal{E}\}$. By a *planar graph isomorphism* on G we mean a continuous, injective function $\omega : |G| \to \mathbb{R}^2$. The injectivity ensures the $\omega(\mathcal{V})$ and $\omega(\mathcal{E})$ are the vertices and edges of some planar graph which we refer to as the image graph, $\omega(G)$.

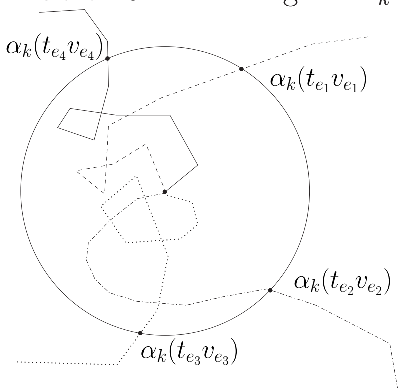
In this section we will perturb the restriction ∂g to get a map ϕ from $\partial\mathbf{D}_T$ to Y_S such that the restriction of ϕ to each $\widetilde{T}$ orbit of a point is a planar graph isomorphism from (the restriction of) $(\mathbf{V}_T, \mathbf{E}_T)$ into the appropriate $\mathbb{R}^2$ orbit of Y_S. This will take two steps, first perturbing ∂g to ϕ_1 and then perturbing ϕ_1 to $\phi_2 = \phi$, the map sought in the Graph Isomorphism Lemma.

5.4. **Near $\mathbf{V}_T$: the first perturbation.** Our first perturbation addresses the issue that it may happen that for some $\widetilde{c} \in \mathbf{V}_T$, that there are distinct points $\widetilde{x} \in e(\widetilde{c}, r)$ and $\widetilde{y} \in e'(\widetilde{c}, r)$ such that $\partial g(\widetilde{x}) = \partial g(\widetilde{y})$. In what follows we will employ the following notation. For $v_e \in \mathbb{R}^2$, let $[0, v_e]$ denote the line segment from 0 to v_e in $\mathbb{R}^2$. For $A \subset \mathbb{R}^2$ and $\widetilde{c} \in X_T$, let $\widetilde{T}^A\widetilde{c} = \{\widetilde{T}^r\widetilde{e} : r \in A\}$. Let $\widetilde{c} \in \mathbf{Q}_k$.

Each edge $e \subset star(\widetilde{c})$ is a set of the form $\widetilde{T}^{[0,v_e]}(\widetilde{c})$ for some $v_e \in \mathbb{R}^2$. The map α_k restricted to $[0, v_e]$ is a piecewise linear map by the definition of g. Let t_e be the largest value of t such that $\|\alpha_k(tv_e))\| = R$. Our choice of R ensures $r \leq \|t_e v_e\| \leq Q$.

Because $\|t_e v_e\| \geq r$, the image points $\alpha_k(t_e v_e)$ are distinct for the different edges $e \subset star(\widetilde{c})$ (due to the choice of r). The situation is accurately reflected in Figure 5.

FIGURE 5. The image of α_k.



We **define** ϕ_1 as follows. For $\widetilde{x} \in \partial \mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some $\widetilde{c} \in \mathbf{Q}_k$, edge $e \subset star(\widetilde{c})$, and $0 \leq t \leq t_e$, define

$$\phi_1(\widetilde{x}) = \phi_1(\widetilde{T}^{tv_e}(\widetilde{c})) = \widetilde{S}^{(t/t_e)\alpha_k(t_e v_e)} \partial g(\widetilde{c}).$$

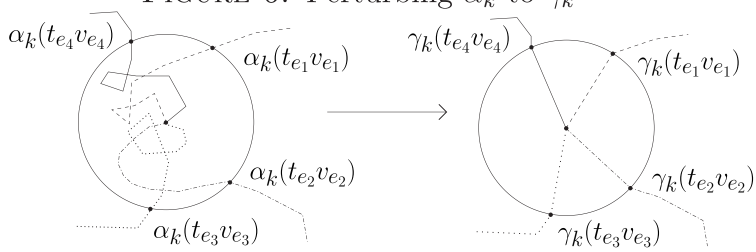
Let $\phi_1(\widetilde{x}) = \partial g(\widetilde{x})$ otherwise (for $\widetilde{x} \in \partial \mathbf{D}_T$ for which there is no such $\widetilde{c}$, e, and t).

We may also regard this as a perturbation of the map α_k. Namely, for $\widetilde{c} \in \mathbf{Q}_k$, $e \in \mathbf{E}_T$ with $e \subset star(\widetilde{c})$, and $0 \leq t \leq t_e$ define

$$\gamma_k(tv_e) = (t/t_e)\alpha_k(t_e v_e).$$

Otherwise, let $\gamma_k(tv_e) = \alpha_k(tv_e)$. Then, for $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ we have $\phi_1(\widetilde{x}) = \widetilde{S}^{\gamma_k(tv_e)} \partial g(\widetilde{c})$. Figure 6 depicts the effect of the perturbation $\alpha_k \mapsto \gamma_k$.

FIGURE 6. Perturbing α_k to γ_k



Claim 27. *$\phi_1 : \partial \mathbf{D}_T \to Y_S$ is well-defined.*

Proof. For each edge $e \in \mathbf{E}_T$ there are precisely two distinct points $\widetilde{c}, \widetilde{c}' \in \mathbf{V}_T$ such that $e \subset star(\widetilde{c})$ and $e \subset star(\widetilde{c}')$. Therefore every point $\widetilde{x} \in e$ is equal to $\widetilde{T}^{tv}(\widetilde{c}) = T^{t'v'}(\widetilde{c}')$

where $t' = 1 - t$, $v' = -v$ and $\widetilde{T}^v(\widetilde{c}) = \widetilde{c}'$. The only one way ϕ_1 could fail to be well-defined would be if at least one of t_e or t'_e is greater than 1/2. However, note that $d_T(\widetilde{c}, \widetilde{c}') = \|v\| \geq m > 2Q$ and t_e and t'_e were chosen such that $0 < t_e\|v\|$, $t'_e\|v'\| \leq Q$, forcing $0 < t_e$, $t'_e < 1/2$. □

Claim 28. *$\phi_1 : \partial\mathbf{D}_T \to Y_S$ is continuous.*

Proof. For each $\widetilde{c} \in \mathbf{S}_k$ and edge $e \subset star(\widetilde{c})$ the values t_e are constant, as are the set of images $\{\alpha_k(t_e v_e) : e \subset star(\widetilde{c})\}$. (A similar map exists for each $\mathbf{S}_k$.) Thus for each $t \in [0,1]$ the map $\partial g(\widetilde{c}) \mapsto \widetilde{S}^{\gamma_k(tv_e)}\partial g(\widetilde{c})$ is achieved by the same map $\widetilde{S}^{\gamma_k(tv_e)}$ for all $\widetilde{c} \in \mathbf{S}_k$, and as a map from $[0,1] \to Y_S$, the map $t \mapsto \widetilde{S}^{\gamma_i(tv_e)}\partial g(\widetilde{c})$ is continuous, thus we have a continuous map from each edge $e \subset \mathbf{E}_T$ to Y_S. These maps agree on the intersections (in $\mathbf{V}_T$), thus we have a continuous map from $\partial\mathbf{D}_T$ to Y_S. □

Claim 29. *For each $\widetilde{c} \in \mathbf{V}_T$ and each edge $e \subset star(\widetilde{c})$,*

$$\phi_1(e) \subset \{\widetilde{s} \in Y_S : d_S(\widetilde{s}, \partial g(e)) \leq R\}.$$

Proof. This should be clear. The only place the image of $\phi(e)$ differs from that of $\partial g(e)$ is prior to the last departure of $\partial g(e)$ from the d_S-R ball about each endpoint of $\partial g(e)$. That portion of $\partial g(e)$ has been modified to stay within the d_S-R ball of the endpoint of $\partial g(e)$, otherwise $\phi(e)$ has the same image as $\partial g(e)$. □

The next lemma states that ϕ_1 is virtually the graph isomorphism ϕ in the Graph Isomorphism Lemma (24). It says that the ϕ_1 images of disjoint edges are disjoint, and if two edges meet at a vertex then their images intersect at the image of that vertex and only at that image. We are verifying that Figure 7 presents an essentially correct picture of the image $\phi_1(\partial\mathbf{D}_T)$ in the $\widetilde{S}$ orbit of a single point.

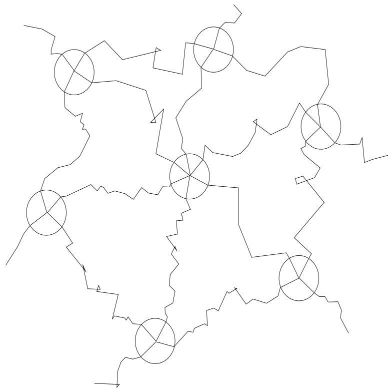


FIGURE 7. The ϕ_1 image of $\partial\mathbf{D}_T$

Lemma 30. *Let e, f be two distinct edges in $\mathbf{E}_T$. Suppose $\widetilde{x} \in e$ and $\widetilde{y} \in f$. Then $\phi_1(\widetilde{x}) = \phi_1(\widetilde{y})$ implies $\widetilde{x} = \widetilde{y}$.*

Proof. We proceed with a series of cases. If e, f are not elements of the same $\mathbb{R}^2$ orbit, then it is clear that $e \cap f = \phi_1(e) \cap \phi_1(f) = \emptyset$.

So, we suppose e and f are in the same $\widetilde{T}$ orbit of X_T. If $e \cap f = \emptyset$ then by our choice of $m > m_Q$ and by the Tiling Lemma 22, part 3d, we have $d_T(e, f) > 2Q$ which means, by Proposition 26, that $d_S(\partial g(e), \partial g(f)) > 2R$. However, by Claim 29, $\phi_1(e)$ is within R of $\partial g(e)$, and $\phi_1(f)$ is within R of $\partial g(f)$, thus $\phi_1(e)$ and $\phi_1(f)$ are disjoint.

Now suppose $e \cap f = \{\widetilde{c}\} \in \mathbf{S}_k \subset \mathbf{V}_T$, then $e, f \subset star(\widetilde{c})$. Every element of e and f can be written as $\widetilde{T}^{t_1 v_e}(\widetilde{c})$ and $\widetilde{T}^{t_2 v_f}(\widetilde{c})$ for some $t_1, t_2 \in [0, 1]$. We want to show that unless $t_1 = t_2 = 0$ their ϕ images, $\phi_1\left(\widetilde{T}^{t_1 v_e}(\widetilde{c})\right)$ and $\phi_1\left(\widetilde{T}^{t_2 v_f}(\widetilde{c})\right)$ are distinct. This is equivalent to showing $\gamma_k(t_1 v_e) \neq \gamma_k(t_2 v_f)$.

First assume $t_1 \in [0, t_e]$ and $t_2 \in [0, t_f]$ are not both zero. Then since γ_k is linear on both $[0, t_e v_e]$ and $[0, t_f v_f]$, $\gamma_k(t_1 v_e) = \gamma_k(t_2 v_f)$ can only happen if $t_1 = t_2 = 0$ or the ray starting at the origin and passing through $\gamma_k(t_e v_e)$ contains $\gamma_k(t_f v_f)$. But $\|\gamma_k(t_e v_e)\| = \|\gamma_k(t_f v_f)\| = R$ by definition, so if $\gamma_k(t_f v_f)$ lies on this ray then $\gamma_k(t_e v_e) = \gamma_k(t_f v_f)$. But one aspect of our choice of t_e and t_f and of our construction of γ_k was that $\gamma_k(t_e v_e) \neq \gamma_k(t_f v_f)$, therefore $\gamma_k(t_1 v_e) \neq \gamma_k(t_2 v_f)$

Now assume $t_1 \in (t_e, 1]$ and $t_2 \in [0, t_f]$. Because t_e is the maximal value of $t \in [0, 1]$ with $\|\gamma_k(t v_e)\| \leq R$, we have $\|\gamma_k(t_1 v_e)\| > R$ but $\|\gamma_k(t_2 v_f)\| \leq R$. A similar result holds for $t_1 \in [0, t_e]$ and $t_2 \in (t_f, 1]$.

Next assume $t_1 \in (t_e, 1]$ and $t_2 \in (t_f, 1]$. The edge intersection $e \cap f = \widetilde{c}$ is a single point. That means the other ends of e and f are distinct elements of $\mathbf{V}_T$, $\widetilde{a} = \widetilde{T}^{v_e}(\widetilde{c}) \in \mathbf{S}_i$, $\widetilde{b} = \widetilde{T}^{v_f}(\widetilde{c}) \in \mathbf{S}_j$ for some $i, j \in 1, .., K$ (so $d_S(\widetilde{a}, \widetilde{b}) \geq m > 2Q$). There exist minimal values t'_e and t'_f such that $t_e < t'_e < 1$ and $t_f < t'_f < 1$ and $\|\gamma_i((t'_e - 1)v_e)\| = R = \|\gamma_j((t'_f - 1)v_f)\|$.

Assume $t_1 \in [t_e, 1 - t'_e]$ and $t_2 \in [t_f, 1 - t'_f]$. For such values of t_1 and t_2, $\widetilde{T}^{t_1 v_e}(\widetilde{c}) \in e(\widetilde{c}, r)$ and $\widetilde{T}^{t_2 v_f}(\widetilde{c}) \in f(\widetilde{c}, r)$ and for such points $\phi_1 = \partial g$. By Proposition 25, $\phi_1 \widetilde{T}^{t_1 v_e}(\widetilde{c}) = \partial g \widetilde{T}^{t_1 v_e}(\widetilde{c}) \neq \partial g \widetilde{T}^{t_2 v_f}(\widetilde{c}) = \phi_1 \widetilde{T}^{t_2 v_f}(\widetilde{c})$.

Finally, assume $t_1 \in [0, 1]$ and $t_2 \in [1 - t'_f, 1]$. Because $\widetilde{b}$ is not an endpoint of e, but is an endpoint of an edge which does not intersect e, it follows that $d_T(e, \widetilde{b}) > 2Q$. It follows by Proposition 26, that $d_S\left(\partial g\left(\widetilde{T}^{t_1 v_e}(\widetilde{c})\right), \partial g\left(\widetilde{T}^{t_2 v_f}(\widetilde{c})\right)\right) > 2R$, and by Claim 29 that $\phi_1\left(\widetilde{T}^{t_1 v_e}(\widetilde{c})\right) \neq \phi_1\left(\widetilde{T}^{t_2 v_f}(\widetilde{c})\right)$. Similar results hold for $t_1 \in [1 - t'_e, 1]$ and $t_2 \in [0, 1]$.

All possibilities for t_1 and t_2 have been exhauseted, so the lemma holds. □

5.5. **Far from vertices: the second perturbation.** In order for ϕ_1 to be the graph homomorphism we want, ϕ_1 must injective on each edge, which at the moment, is not necessarily the case. However, this is easily remedied. To do so we will focus on the γ_k.

We have constructed γ_k so that the following holds. For $\widetilde{x} \in \partial\mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some k, $\widetilde{c} \in \mathbf{Q}_k$, $e \in \mathbf{E}_T$, $e \subset star(\widetilde{c})$, and $0 \leq t \leq 1$,

$$\phi_1(\widetilde{x}) = \widetilde{S}^{\gamma_k(tv_e)}\partial g(\widetilde{c}).$$

The map $t \mapsto \gamma_k(tv_e)$ is a piecewise linear function from $[0,1]$ to $\mathbb{R}^2$ which is not necessarily injective. We now refer to the following.

Lemma 31. *Let $H : [0,1] \to \mathbb{R}^2$ be a piecewise linear function. There exists an injective piecewise linear function $F : [0,1] \to \mathbb{R}^2$ such that $F(0) = H(0)$, $F(1) = H(1)$ and $F([0,1]) \subset H([0,1])$.*

Working with one map of the form $t \mapsto \gamma_k(tv_e)$ at a time (and there are only finitely many of these maps), we apply the above lemma to each γ_k with each edge. We obtain a finite set of injective maps $t \mapsto \widehat{\gamma}_k(tv_e)$ which are also continuous and piecewise linear and have the same image. For $\widetilde{x} \in \partial\mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some k, $\widetilde{c} \in \mathbf{Q}_k$, $e \in \mathbf{E}_T$, $e \subset star(\widetilde{c})$ and $0 \leq t \leq 1$, we define ϕ as follows

$$\phi(\widetilde{x}) = \widetilde{S}^{\widehat{\gamma}_k(tv_e)}\partial g(\widetilde{c}).$$

Thus we have a continuous injective, orbit injective mapping $\phi : \partial\mathbf{D}_T$ into Y_S and therefore a graph injection $\phi : (\mathbf{V}_T, \mathbf{E}_T) \to Y_S$, proving the Graph Isomorphism Lemma (24).

6. Extending Graph Isomorphisms

Given the map $\phi : \partial\mathbf{D}_T \to Y_S$ from the Graph Isomorphism Lemma, we now wish to extend it to a continuous injection $\psi : X_T \to Y_S$. Let $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \{\widehat{\mathbf{V}}_i\}, \{D_i\})$ be the tiling of X_T by action polygons from the Tiling Lemma. First, for each $\widehat{\mathbf{V}}_i$ we identify the sets $\widehat{\mathbf{U}}_i^j$ on which the map $\phi : \widetilde{T}^{\partial D_i}\widehat{\mathbf{U}}_i^j \to Y_S$ has a fixed cocycle $\eta_i^j : \partial D_i \to \mathbb{R}^2$. Then that cocycle is extended to $\bar{\eta}_i^j : D_i \to \mathbb{R}^2$ and used to construct ψ. The injectivity follows from Lemma 32.

For each $\widetilde{x} \in \widehat{\mathbf{V}}_i$ and $v \in \partial D_i$, let $\eta(\widetilde{x}, v) \in \mathbb{R}^2$ be such that $\phi(\widetilde{T}^v\widetilde{x}) = S^{\eta(\widetilde{x},v)}\phi(\widetilde{x})$. (For each $\widetilde{x} \in \widehat{\mathbf{V}}_i$, $\eta(\widetilde{x}, \cdot) : \partial D_i \to \mathbb{R}^2$ is a piecewise linear homeomorphism.) Let $\mathcal{V}(D)$ be the set of extreme points of the polygon D_i. For each $\widetilde{x} \in \widehat{\mathbf{V}}_i$ and $v \in \mathcal{V}(D_i)$, the element $\widetilde{T}^v\widetilde{x} \in \mathbf{V}_T$, and consequently there is a k (depending on $\widetilde{x}$) such that $\widetilde{T}^v\widetilde{x} \in \mathbf{Q}_k$. Let $K(v,i)$ be the collection of all such k for $\widetilde{x} \in \widehat{\mathbf{V}}_i$ and $v \in \mathcal{V}(D_i)$. So, the (non-empty) sets of the form $\widehat{\mathbf{U}}_i^j = \widehat{\mathbf{V}}_i \cap \{\widetilde{T}^{-v}\mathbf{Q}_k : k \in K(v,i) \text{ and } v \in \mathcal{V}(D_i)\}$ form a clopen (sub)partition of $\widehat{\mathbf{V}}_i$. The map $\eta(\widetilde{x}, \cdot) : \partial D_i \to \mathbb{R}^2$ is constant for all $\widetilde{x} \in \widehat{\mathbf{U}}_i^j$, thus we regard $\eta(\widetilde{x}, \cdot)$ as a collection of maps $\{\eta_i^j\}$ where $\eta_i^j : \partial D_i \to \mathbb{R}^2$.

As such, we can extend each η_i^j to a homeomorphism on all of D_i, $\bar{\eta}_i^j : D_i \to \mathbb{R}^2$. We are now able to define the map $\psi : X_T \to Y_S$. For each $\widetilde{x} \in X_T$ there is a $\widehat{\mathbf{U}}_i^j$ such that $\widetilde{x} \in \widetilde{T}^{D_i}\widehat{\mathbf{U}}_i^j$ (if $\widetilde{x} \in \partial D_T$ there will be more than one such $\widehat{\mathbf{U}}_i^j$). For each such $\widehat{\mathbf{U}}_i^j$

there exits a unique $u \in D_i$ such that $\widetilde{T}^{-u}\widetilde{x} \in \widehat{\mathbf{U}}_i^j$ (the uniqueness is assured just as it is ensured in Part 3 of Lemma 21).

We would like to define $\psi(\widetilde{x})$ by letting $\widetilde{c} = \widetilde{T}^{-u}\widetilde{x}$ (an element of $\widehat{V}_i$), and defining

$$\psi(\widetilde{x}) = S^{\bar{\eta}_i^j(u)}\psi(\widetilde{c}),$$

but that would require $\psi(\widetilde{c})$ to be defined, which it isn't. So we must be slightly less direct. We must define ψ in terms of the elements in ∂D_i, where ϕ is defined.

For $\widetilde{x} \in X_T$, $\widetilde{c} \in \widehat{\mathbf{U}}_i^j$, and $u \in D_i$ such that $\widetilde{x} = \widetilde{T}^u\widetilde{c}$, and for $v \in \mathcal{V}(D_i)$ and $k \in K(v,i)$ such that $\widetilde{T}^v\widetilde{c} \in \mathbf{Q}_k \subset \mathbf{V}_T$, define

$$\psi(\widetilde{x}) = S^{\bar{\eta}_i^j(u)}S^{-\bar{\eta}_i^j(v)}\phi(\widetilde{T}^v\widetilde{c}).$$

One can check that ψ is a well-defined map. By construction it is continuous. It remains to show that ψ is an injection.

By construction ψ is an orbit injection, and we must check that ψ is injective on each orbit. We will make this argument by considering the behavior of graph isomorphisms between graphs in the *plane*. So, let take a moment to map our graphs and their isomorphisms to $\mathbb{R}^2$. We ae essectially selecting a connected subset of $\Pi_T^{-1}(\partial D_T)$ in $\{x\} \times \mathbb{R}^2$ for some $x \in X$ and locating the graph that covers it.

For $\widetilde{x}, \widetilde{y} \in X_T$ in the same $\widetilde{T}$ orbit, let $v(\widetilde{y}, \widetilde{x}) \in \mathbb{R}^2$ be the element of $\mathbb{R}^2$ for which $\widetilde{T}^{v(\widetilde{y},\widetilde{x})}\widetilde{x} = \widetilde{y}$. A similar map exists in Y_S. The map v is bijective in each component individually. For a set $A \subset X_T$ in the same $\widetilde{T}$ orbit of $\widetilde{x}$, let $v(A, \widetilde{x}) = \{v(a, \widetilde{x}) : a \in A\}$, and for general $A \subset X_T$, let $v(A, \widetilde{x}) = v(A \cap \widetilde{T}^{\mathbb{R}^2}\widetilde{x}, \widetilde{x})$ where $\widetilde{T}^{\mathbb{R}^2}\widetilde{x}$ is the $\widetilde{T}$ orbit of $\widetilde{x}$. So, $v(A, \widetilde{x})$ is the subset of $\mathbb{R}^2$ that maps $\widetilde{x}$ to the portion of A in the $\widetilde{T}$ orbit of $\widetilde{x}$. For each $\widetilde{x} \in X_T$ we will use $v(\cdot, \widetilde{x})$ to map subsets of the $\widetilde{T}$ orbit of $\widetilde{x}$ bijectively into subsets of $\mathbb{R}^2$. In the graph $G(V_T, E_T)$, E_T is a collection of subset of X_T; we write $v(E_T, \widetilde{x})$ to mean the collection of maps of those edges in the $\widetilde{T}$ orbit of $\widetilde{x}$. Then

$$(v(V_T, \widetilde{x}), v(E_T, \widetilde{x}))$$

is a graph embedded in $\mathbb{R}^2$. One can check that $(v(V_T, \widetilde{x}), v(E_T, \widetilde{x}))$ is the graph that tiles the boundaries $\{u + \partial D_{i(u)} : \widetilde{T}^u\widetilde{x} \in \widetilde{V}_T\}$.

By the Graph Isomorphism Lemma the map $\phi : \partial D_T \to Y_S$ applied to (V_T, E_T) is a graph isomorphism. So $(\phi(V_T)), \phi(E_T))$ is a graph in Y_S, and for $\phi(\widetilde{x})$, the map $(v(\phi(V_T), \phi(\widetilde{x})), v(\phi(E_T), \phi(\widetilde{x}))$ is a graph in $\mathbb{R}^2$. As a composition the map

$$\Phi_{\widetilde{x}} : (v(V_T, \widetilde{x}), v(E_T, \widetilde{x})) \mapsto (v(\phi(V_T), \phi(\widetilde{x})), v(\phi(E_T), \phi(\widetilde{x})))$$

is a graph isomorphism between graphs embedded in $\mathbb{R}^2$.

For $\widetilde{x} \in X_T$, for each $u \in \mathbb{R}^2$ such that $\widetilde{T}^u\widetilde{x} \in \widehat{\mathbf{V}}_T$ there is a unique $i(u)$ such that $\widetilde{T}^u\widetilde{x} \in \widehat{\mathbf{V}}_{i(u)}$. The collection $\{u + D_{i(u)} : \widetilde{T}^u\widetilde{x} \in \widehat{\mathbf{V}}_T\}$ is a finite tiling of $\mathbb{R}^2$, whose boundary $\widetilde{T}$ maps onto the portion of the graph $(\mathbf{V}_T, \mathbf{E}_T)$ in the $\widetilde{T}$ orbit of $\widetilde{x}$.

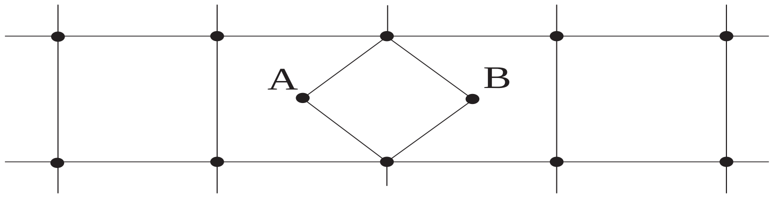


FIGURE 8. A graph whose graph isomorphism does not extend to $\mathbb{R}^2$.

Because the edges $v(E_T, \widetilde{x})$ tile $\{u + D_{u(i)} : \widetilde{T}^u\widetilde{x} \in \widehat{\mathbf{V}}_T\}$ we can regard $\Phi_{\widetilde{x}}$ as a map from each $u + D_{i(u)}$ to $\mathbb{R}^2$. We know the $\Phi_{\widetilde{x}}$ image of each $u + D_{i(u)}$ (for $\widetilde{T}^u\widetilde{x} \in \widehat{\mathbf{V}}_T$) is a polygon because $\Phi_{\widetilde{x}}$ is a piecewise linear map on $u + \partial D_{i(u)}$. Moreover, because the $\widehat{\mathbf{U}}_i^j$ are a (finite) clopen cover of the compact set $\widehat{\mathbf{V}}_T$, and because for u, u' such that $\widetilde{T}^u\widetilde{x}$, $\widetilde{T}^{u'}\widetilde{x} \in \widehat{\mathbf{U}}_i^j$, the images $\Phi_{\widetilde{x}}(u + D_{i(u)})$ and $\Phi_{\widetilde{x}'}(u' + D_{i(u')})$ are the same up to translation, so, up to translation, there are only a finite number of image polygons in the collection

$$\left\{\Phi_{\widetilde{x}}(u + D_{i(u)}) : \widetilde{T}^u\widetilde{u} \in \widehat{\mathbf{V}}_T\right\}$$

For a finite tiling $\mathcal{D}$ of the plane by convex polygons whose boundary is tiled by the edges of the graph $G = (\mathcal{V}, \mathcal{E})$ and a graph isomorphism $\alpha : (\mathcal{V}, \mathcal{E}) \to \mathbb{R}^2$, let us define the following extension. For each tile $D \in \mathcal{D}$ let $\beta_D : D \to \mathbb{R}^2$ be an extension of α restricted to ∂D. For $x \in \mathbb{R}^2$, define $\beta(x) = \beta_D(x)$ for $D \in \mathcal{D}$ containing x. Let us call β a *natural tiling extension* of α. Notice that a tiling extension β need not be either injective or surjective even though α was injective. See also Figure 8.

The extension ψ, defined earlier, induces a natural tiling extension of the graph isomorphism $\Phi_{\widetilde{x}}$ to form a map $\Psi_{\widetilde{x}} : \mathbb{R}^2 \to \mathbb{R}^2$. Showing that the map ψ is injective is equivalent to showing that $\Psi_{\widetilde{x}}$ is injective.

What makes our situation simple is that tiles in a Delaunay tiling are convex and for $\widetilde{x} \in X_T$, both sets $\{v : \widetilde{T}^v(\widetilde{x}) \in \mathbf{V}_T\}$ and $\{v : \widetilde{S}^v(\phi(\widetilde{x})) \in \phi(\mathbf{V}_T)\}$ are *uniformly discrete* subsets of $\mathbb{R}^2$. A set $U \subset \mathbb{R}^2$ is uniformly discrete if there is an $n > 0$ such that $u, v \in U$ implies that $u + B(0, n)$ and $v + B(0, n)$ are disjoint.

If, for example, tiles in the domain are not convex, there can be a counter-example as is demonstrated by the example depicted in Figure 8. Consider the graph isomorphism which is the identity on all the vertices except for A and B; the isomorphism exchanges them. The edges are mapped by extension of the the vertex map. Here we see 2-cells bounding tiling polygons whose images contain vertices. One can also construct counterexamples when the image of the set of vertices is not discrete. For example, one could map the entire plane inside a single tile boundary.

The extension ψ is injective if we can prove the following about the graph isomorphisms. General results of this type are surely known. For completeness we include a proof.

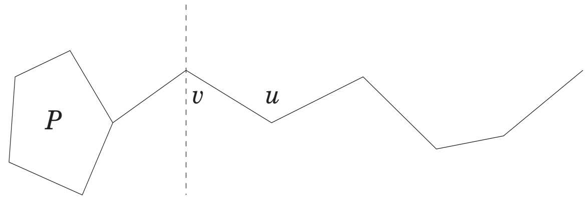


FIGURE 9. The cardinality of $\partial\mathbf{T}\backslash P$ is not finite.

Lemma 32. *Let $\mathcal{D}$ be a finite tiling of $\mathbb{R}^2$ by convex polygons, and let $(\mathcal{V},\mathcal{E})$ be the graph corresponding to $\partial\mathcal{D}$. Suppose $\alpha : \partial\mathcal{D} \to \mathbb{R}^2$ is a graph isomorphism such that $\alpha(\mathcal{V})$ is uniformly discrete. Then the α-image of every 2-cell in $\mathcal{D}$ is a 2-cell.*

Proof. Let P be a polygonal tile and consider any vertex $v \in \partial\mathcal{D}\backslash\partial P$. We first consider the possibility that $\alpha(v)$ is in the interior of $\alpha(\partial P)$. Because every tile is convex, it not hard to see that the vertex cardinality of the graph component in $\partial\mathcal{D}\backslash\partial P$ containing v is countably infinite. To wit, there exists a half-plane containing P whose boundary contains v. The tile convexity property implies the complementary half-plane contains an edge e which ends at v (so $e = [u, v]$). Repeating this argument at u we construct an infinite sequence of edges which don't intersect P. (See Figure 9.)

Because α is injective, the α image of any non-self-intersecting loop is also a non-self-intersecting loop. The continuity of α ensures the image is a bounded loop. The Jordan Curve Theorem tells us the image bounds a bounded set (which we refer to as the *interior*) and that the image curve together with the interior is homeomorphic to a disk. If $\phi(v)$ is contained in the interior of $\alpha(\partial P)$, then the interior of $\alpha(\partial P)$ must also contain the α image of the countable connected component of v. That is, the interior contains a countable number of vertices. This contradicts the uniform discreteness of the image $\alpha(\mathcal{V})$ and the boundedness of the interior. Thus, the interior $\alpha(\partial P)$ does not contain the α image of any vertices.

Next we consider the possibility that an edge in $\partial\mathcal{D}$ has an image in the interior of $\alpha(\partial P)$. If so, then the image of that edge begins and ends at vertices in $\alpha(\partial P)$ or else we are in the case considered previously. For a set of edges $\{e_1, e_2, .., e_n\}$, let $\mathcal{V}(\{e_1, e_2, .., e_n\}) = \{v \in \mathcal{V} : v \in e_i \text{ for some } 1 \le i \le n\}$, i.e., the set of vertices contained in $\mathcal{E}$. Let Γ and Γ' be non-self-intersecting loops of edges bounding interiors P and P'. If $\mathcal{V}(\Gamma') \subsetneqq \mathcal{V}(\Gamma)$ and if P and P' are convex, then $P' \subsetneqq P$. So, if the interior of the region bounded by $\alpha(\partial P)$ does intersect an edge $\alpha(e')$ for some edge, $e' \in \mathcal{V}(\Gamma')$, then we can construct Γ' with $\mathcal{V}(\Gamma') \subsetneqq \mathcal{V}(\Gamma)$ where Γ is the list of edges bounding P. The curve Γ' is non-self-intersecting and bounds a convex region P', hence $P' \subsetneqq P$, contradicting the property that P is a tile.

Therefore, for each tile $P \in \mathcal{D}$, its boundary ∂P is a 2-cell which α then takes to the 2-cell $\alpha(\partial P)$. Both P and the interior of the image loop $\alpha(\partial P)$ are homeomorphic to disks, thus there exists a homeomorphic extension β from P to the interior of $\alpha(\partial P)$. Thus, because α maps distinct 2-cells to distinct 2-cells and $\mathcal{D}$ tiles the plane, α can be extended to a continuous injection $\beta : \mathbb{R}^2 \to \mathbb{R}^2$. □

6.1. **Surjectivity and 2nd Category Sets.**

Lemma 33. *The map ψ is surjective.*

Proof. Because ψ is continuous and X_T is compact, the image $\psi(X_T) \subset Y_S$ is closed. If $\psi(X_T)$ is shift invariant, then $\psi(X_T) = Y_S$. Since Y_S is minimal, we will show $\psi(X_T)$ is shift invariant.

Let $\widetilde{y} \in \Phi(X_T)$, and suppose by way of contradiction that $\widetilde{S}^v(\widetilde{y})$ is not in $\Phi(X_T)$. Consider the set $[\widetilde{y}, \widetilde{S}^v(\widetilde{y})] = \{\widetilde{S}^{tv}(\widetilde{y}) : t \in [0,1]\} \subset Y_S$. Set $t_* = \max\{t \in [0,1] : \widetilde{S}^{tv}(\widetilde{y}) \in \Phi(X_T)\}$ and $\widetilde{z} = \widetilde{S}^{t_* v}(\widetilde{y})$. Then $\widetilde{z} = \Phi(\widetilde{x})$ for $\widetilde{x} \in X_T$. Because ψ is a continuous injective map, for any $\epsilon > 0$ there is a $\delta > 0$ sufficiently small such that $\{\widetilde{T}^w(\widetilde{x}) : \|w\| = \delta\}$ maps to a loop around $\widetilde{z}$ which lies inside $\{\widetilde{S}^w(\widetilde{z}) : \|w\| < \epsilon\}$. If $\epsilon > 0$ is sufficiently small, this latter set must intersect $[\widetilde{y}, \widetilde{S}^v(\widetilde{y})]$ at a point $\widetilde{S}^{tv}(\widetilde{y})$ where $t_* < t < 1$, a contradiction. □

Now we return to address the proof of Theorem 7. We will show that $h(C)$ (where $C \subset X$ is from Lemma 20) contains an open set.

Let us say that $h(X)$ *occurs syndetically in* Y (with respect to S) if $\bigcup_{\|v\|<n} S^v h(X) = Y$ for some $n > 0$. For $C \subset X$ clopen, we say that the image $h(C)$ *occurs syndetically in its own orbit* if there exists $n > 0$ such that for any $w \in \mathbb{Z}^2$ and $y \in h(C)$, there exists $v \in \mathbb{Z}^2$ such that $\|v\| < n$ and $S^{w-v}y \in h(C)$. That is, there exists an $n > 0$ such that

$$\bigcup_{\|v\|<n} S^v h(C) = \bigcup_{v\in\mathbb{Z}^2} S^v h(C). \tag{6.1}$$

Lemma 34. *Let (X,T) be a minimal $\mathbb{Z}^2$ action on a Cantor set and let $C \subset X$ be clopen. The following are equivalent.*

1. *$h(C)$ contains an open set in Y,*
2. *$h(C)$ occurs syndetically in Y,*
3. *the image $h(C)$ occurs syndetically in its own orbit.*

Proof. (1) implies (2) because S is minimal. (2) implies (3) is clear. Assume (3), that $h(C)$ occurs syndetically in its own orbit. The set $h(C)$ is closed and therefore $\bigcup_{v\in\mathbb{Z}^2} S^v h(C) = \bigcup_{\|v\|<n} S^v h(C)$ is closed. But every S orbit is dense which means that $\bigcup_{v\in\mathbb{Z}^2} S^v h(C)$ is dense and closed in Y, i.e., $\bigcup_{v\in\mathbb{Z}^2} S^v h(C) = Y$. If $h(C)$ contains no open sets, then $h(C)$ is nowhere dense in Y, and the same is true of $S^v h(C)$ for any $v \in \mathbb{Z}^2$. This would mean that Y is a countable union of nowhere dense sets, contradicting the Baire Category Theorem. □

Proof of Theorem 7. We note $(X,T) \simeq \left(\Pi_T(X \times \{0\}), \widetilde{T}\big|_{\mathbb{Z}^2}\right)$, and $\psi(\Pi_T(C \times \{0\})) = \Pi_S(h(C) \times \{0\}) \simeq h(C)$. By construction ψ maps $\Pi_T(C \times \{0\}) = \mathbf{V}_T$ into an image syndetic in its orbit. By Lemma 33 this same image is syndetic in $(\Pi_S(Y \times \{0\}), \widetilde{S}\big|_{\mathbb{Z}^2}) \simeq (Y,S)$ and thus by the Lemma 34 contains an open set. ■

7. The Obstruction to Fact 16 for $d \geq 3$.

Given $k > 0$ and for $n >> k$, let $\mathcal{V} = (n\mathbb{Z})^3$. $\mathcal{V}$ is n-regular. The Delaunay cells of $\mathcal{V}$ are the cubes of length n on a side with corners in $\mathcal{V}$. We perturb $\mathcal{V}$ by letting $\mathcal{V}' = \{(0,0,k)\} \cup (\mathcal{V} \backslash (0,0,0))$. That is, by shifting the origin up k units. The eight Delaunay cells containing $(0,0,0)$ splinter into 24 new Delaunay cells.

To see this, scale by n and consider $\mathcal{V} \subset \mathbb{Z}^3 \subset \mathbb{R}^3$. Let $\mathcal{V}' = \{(0,0,\epsilon)\} \cup (\mathcal{V} \backslash (0,0,0))$ for $\epsilon > 0$ sufficiently small. Focus on the Delaunay cells $D^{\pm}$ from $\mathcal{V}$ with extreme points

$$\mathcal{V}(D^{\pm}) = \{(0,0,0),\ (0,1,0),\ (1,0,0),\ (0,0,\pm1),\ (1,1,0),\ (0,1,\pm1),\ (1,0,\pm1),\ (1,1,\pm1)\}.$$

The reader can show that for $\mathcal{V}'$ there are now 6 Delaunay tiles $\{D'_i\}_{i=1}^6$ with $D^+ \cup D^- = D'_1 \cup \cdots \cup D'_6$. The D'_i are the convex polyhedra with the following extreme point sets. $\{(0,0,\epsilon),\ (1,0,0),\ (1,1,0),\ (1,0,1),\ (1,1,1)\}$, $\{(0,0,\epsilon),\ (0,1,0),\ (0,1,1),\ (1,1,0),\ (1,1,1)\}$, $\{(0,0,\epsilon),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,1,1)\}$, $\{(0,0,\epsilon),\ (1,0,0),\ (0,1,0),\ (0,0,-1)\}$, $\{(0,0,-1),\ (1,0,0),\ (1,1,0),\ (0,1,0),\ (0,1,-1),\ (1,0,-1),\ (1,1,-1)\}$, and $\{(0,0,\epsilon),\ (1,0,0),\ (1,1,0),\ (0,1,0)\}$.

It is the last polyhedron that is problematic. The edges $[(1,0,0),\ (0,1,0)]$ and $[(0,0,\epsilon),\ (1,1,0)]$ pass within ϵ of each other. Scaled up, this means there are n-regular subsets in $\mathbb{Z}^3$ with Delaunay polyhedra which have edges that pass within 1 unit at their centers while their ends are at least $n-1$ separated.

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