Morphisms from Non-periodic $\mathbb{Z}^2$ Subshifts I: Constructing Embeddings from Homomorphisms

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Abstract. In this paper we introduce foundational techniques and prove the following: If X is a $\mathbb{Z}^d$ subshift without periodic points, if Y is a $\mathbb{Z}^d$ square mixing SFT containing a finite orbit and if there exists a homomorphism $X \to Y$, then X embeds into Y if and only if $h(X) < h(Y)$. For the proof, clopen markers are used to generate Voronoi tiles whose thickened boundaries are coded using the homomorphism. The entropy gap and the square mixing permit the construction of an injective code on the tile interiors. A sequel will show $\mathbb{Z}^2$ square filling mixing shifts of finite type are square mixing and that homomorphisms exist, resulting in an extension of Krieger's Embedding Theorem to $\mathbb{Z}^2$ subshifts.

1. Introduction

The theory of $\mathbb{Z}$ subshifts of finite type (SFTs) is a rich, well understood theory with many applications [LM, DGS, Bow]. It has developed in parallel with ergodic theory and there have been frequent connections, such as, for example, the Jewett-Krieger Theorem. A key to such ergodic theory applications is the diverse collection of subsystems found in a mixing SFT of positive entropy. This diversity is demonstrated in the Krieger Embedding Theorem [Kr] which states that given the necessary conditions on entropy and periodic points, any subshift can be embedded in a mixing SFT. Because it so effectively captures the subshifts embeddable in a mixing SFT, Krieger's Embedding Theorem is a cornerstone in the coding theory of $\mathbb{Z}$ SFTs where it finds application in the construction of factor maps [Boy] and automorphisms [BF, KRW] and even has applications in the theory of non-negative matrices [BH].

Because all $\mathbb{Z}$ mixing subshifts of finite type (containing more than one point) have roughly the same structure of subsystems, equilibrium states and lower entropy factors, the mixing SFTs have proven to be a very effective point from which to view $\mathbb{Z}$ subshifts. However such a viewpoint has not been located for $\mathbb{Z}^d$ SFTs for $d \geq 2$, though there are certainly positions offering stunning vistas such as the class of algebraic actions on compact abelian groups (see *e.g.* [S]) or the work on two dimensional tiling systems (see *e.g.* [Ra]). Unfortunately, despite the rich and intricate landscapes displayed in these views, they do not offer the sort of panoramic perspective provided by $\mathbb{Z}$ mixing SFTs.

This is the first in a series of papers which develops a theory of coding for a class of $\mathbb{Z}^2$ SFTs that is similar to the theory available for $\mathbb{Z}$ mixing SFTs. While it may be some time before the appropriate class of target subshifts is precisely determined, a recent example of Sahin and Quas [QS] indicates that it is not the mixing SFTs; they exhibit a $\mathbb{Z}^2$ mixing SFT X and $\epsilon > 0$ which contains no proper square mixing SFTs whose entropy exceeds $h(X) - \epsilon$.

We begin by addressing the case where the domain is a non-periodic subshift. The first two papers will address the construction of homomorphisms and embeddings from non-periodic subshifts, and will prove an extension of Krieger's Embedding Theorem which takes non-periodic subshifts as the domain (later papers will extend this to arbitrary subshifts). We stress, the non-periodic subshifts are not an isolated class of subshifts. By generalizing the Jewett-Krieger Theorem, A. Rosenthal [Ro] has shown that every free ergodic measure preserving $\mathbb{Z}^2$ action on a Lebesgue space is measurably isomorphic to a strictly ergodic $\mathbb{Z}^2$ subshift providing us with an immense supply of non-periodic $\mathbb{Z}^2$ subshifts. Moreover, this restriction of the domain to non-periodic subshifts is not merely an academic exercise but in fact will develop foundational techniques necessary for the general situation.

To state the results of the first paper let $\Lambda_n = \{(x_1, .., x_d) \in \mathbb{Z}^d : |x_i| < n\}$. A $\mathbb{Z}^d$ subshift Z is said to be square mixing (SM) if there exists a $k \geq 0$ such that for $n > 0$ and any pair $x, y \in Z$ there exists $z \in Z$ such that $z|_{\Lambda_n} = y|_{\Lambda_n}$ and $z|_{\mathbb{Z}^d \setminus \Lambda_{n+k}} = x|_{\mathbb{Z}^d \setminus \Lambda_{n+k}}$. We prove the following result: If X is a non-periodic $\mathbb{Z}^d$ subshift, Z is a $\mathbb{Z}^d$ square mixing SFT which contains a finite orbit and there exists a homomorphism $X \to Z$, then there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$, where h denotes the $\mathbb{Z}^d$ entropy.

In a second paper [L] the construction of homomorphisms $X \to Z$ will be discussed. This is a more delicate and difficult construction; it will be necessary to suppose Z is a $\mathbb{Z}^2$ square filling mixing SFT. Roughly speaking, a SFT is square filling if a word which is locally allowed on a square annulus may be filled in. (See [L] for a precise definition.) It will be shown that a square filling mixing SFT is square mixing (and contains a finite orbit) and it will be shown if X is non-periodic and Z is a square filling mixing SFT then there exists a homomorphism $X \to Z$. Combining the results of the two papers we arrive at the following extension of Krieger's Embedding Theorem: If X is non-periodic and Z is a square filling mixing SFT then there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$. We should also stress that this class of square filling SFTs is no mere replica of $\mathbb{Z}$ mixing SFTs. Indeed, the $\mathbb{Z}^2$ strongly irreducible SFT example of Burton and Steif [BS] which has two measures of maximal entropy is a square filling mixing SFT.

The paper will proceed as follows. In Section 2 we present definitions and statements of the results and an overview of the proof of the primary result (Theorem 2.5). Sections 3-7 contain preliminary results (described more thoroughly in Section 2) which are brought together to construct the desired embedding in Section 8. In Section 9 we show $\mathbb{Z}^2$ square mixing SFTs have finite orbits (this is also addressed in [W]) and we show (for $d \geq 2$) $\mathbb{Z}^d$ square mixing SFTs are entropy minimal (Z is entropy minimal if $X \subsetneq Z \;\Rightarrow\; h(X) < h(Z)$).

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2. Definitions and Results.

For an integer $d > 0$ and $x = (x_1, .., x_d) \in \mathbb{R}^d$ let $\|x\|$ refer to the Euclidean norm of x, and for two points $x, y \in \mathbb{R}^d$ let $\rho(x, y) \equiv \|x - y\|$. Let $\|x\|_{\sup} \equiv \max\{|x_i| : 1 \leq i \leq d\}$ and for $k \in \mathbb{N}$ and $v \in \mathbb{R}^d$, let $\Lambda_k + v \equiv \{t \in \mathbb{R}^d : \|t - v\|_{\sup} < k\}$ (writing $\Lambda_k \equiv \Lambda_k + 0$ and $\Lambda_k \backslash 0 \equiv \Lambda_k \backslash \{0\}$). For $r > 0$ and $v \in \mathbb{R}^d$ let $B(v, r) \equiv \{t \in \mathbb{R}^d : \rho(v, t) < r\}$ and for $A \subset \mathbb{R}^2$ let $\overline{A}$ denote the closure of A with respect to the topology induced by the metric ρ. So $\overline{B}(v, r) \equiv \{t \in \mathbb{R}^d : \rho(v, t) \leq r\}$. We will regard $\mathbb{Z}^d$ as naturally embedded in $\mathbb{R}^d$ thus inheriting the above distance functions and sets by restriction.

Let S be a finite set of symbols and let $S^{\mathbb{Z}^d}$ be the set of maps from $\mathbb{Z}^d$ to S with the product topology. The group $\mathbb{Z}^d$ acts naturally on $S^{\mathbb{Z}^d}$ and this action is referred to as the shift on $S^{\mathbb{Z}^d}$. Specifically, an element $v \in \mathbb{Z}^d$ takes $x \in S^{\mathbb{Z}^d}$ to the element $\sigma^v x \in S^{\mathbb{Z}^d}$, where $(\sigma^v x)_u = x_{v+u}$, $\forall u \in \mathbb{Z}^d$. The map σ^v acts homeomorphically on the set $S^{\mathbb{Z}^d}$ with respect to the product topology. A closed shift invariant subset $X \subset S^{\mathbb{Z}^d}$ is called a $\mathbb{Z}^d$ *subshift.* For X and Y subshifts, a continuous map $\phi : X \to Y$ which commutes with the shifts on X and Y is called a *homomorphism.* An injective (bijective) homomorphism is called an *embedding (topological conjugacy).* We say an embedding $\phi : X \hookrightarrow Y$ is *proper* if $\phi(X) \neq Y$. A $\mathbb{Z}^d$ subshift X is *mixing* if for any pair of nonempty open sets U and V, we have $U \cap \sigma^v V \neq \emptyset$ for all but finitely many $v \in \mathbb{Z}^d$.

For $\mathcal{A} \subset \mathbb{Z}^d$ and $x \in S^{\mathbb{Z}^d}$ let $x|_{\mathcal{A}}$ be the element of $S^{\mathcal{A}}$ formed by the restriction of x to $\mathcal{A}$. For $a \in \mathbb{Z}^d$ we use the simpler notation $x_a = x|_{\{a\}}$. For a $\mathbb{Z}^d$ subshift $X \subset S^{\mathbb{Z}^d}$ let $W_X(\mathcal{A}) \equiv \{w \in S^{\mathcal{A}} : \; \exists \; x \in X \text{ such that } x|_{\mathcal{A}} = w\}$ which we call the set of X *allowed* words on $\mathcal{A}$. If $\mathcal{B} \subset \mathbb{R}^d$ we let $x|_{\mathcal{B}} \equiv x|_{\mathcal{B} \cap \mathbb{Z}^d}$ and $W_X(\mathcal{B}) \equiv W_X(\mathcal{B} \cap \mathbb{Z}^d)$. If the cardinality of $\mathcal{A} \subset \mathbb{Z}^d$ is finite (written $|\mathcal{A}| < \infty$) then we say $w \in W_X(\mathcal{A})$ is a *finite* word, otherwise we say w is an *infinite* word. The property of mixing may be expressed with finite words: for $\mathcal{A} \subset \mathbb{Z}^d$ finite, for $\alpha, \beta \in W_X(\mathcal{A})$, and for all but finitely many $v \in \mathbb{Z}^d$ there exists $x \in X$ such that $x|_{\mathcal{A}} = \alpha$ and $(\sigma^v x)|_{\mathcal{A}} = \beta$ (we say x *exhibits* α and β separated by v).

It will be useful to have a notion of a shift on words. For $\mathcal{A} \subset \mathbb{Z}^d$ and $v \in \mathbb{Z}^d$, define $\sigma^v(\mathcal{A}) \equiv \{a - v : a \in \mathcal{A}\} = \mathcal{A} - v$. For $w \in S^{\mathcal{A}}$ and $v \in \mathbb{Z}^d$, let us define $\sigma^v w \in S^{\sigma^v \mathcal{A}}$ as $(\sigma^v w)_u \equiv w_{u+v}, \; \forall \, u \in \sigma^v \mathcal{A}$. This definition is consistent with the shift on $S^{\mathbb{Z}^d}$ and there is the following convenient relation $\sigma^v(x|_{\mathcal{A}}) = \sigma^v x|_{\sigma^v \mathcal{A}}$. As a matter of practical application, if $\mathcal{A} \neq \sigma^v \mathcal{A}$, then $S^{\mathcal{A}}$ and $S^{\sigma^v \mathcal{A}}$ are distinct, which means that $W_X(\mathcal{A})$ and $W_X(\sigma^v \mathcal{A})$ are distinct, nonetheless when $\mathcal{A}$ is finite we will not hesitate to identify their elements.

We will use the following standard definition for the $\mathbb{Z}^d$ entropy of a $\mathbb{Z}^d$ subshift X (where $|\Lambda| = card\{\Lambda\}$ and $\Lambda_n^+ \equiv \{(r_1, .., r_d) : 0 \leq r_i < n \text{ for } 1 \leq i \leq d\}$).

Definition 2.1.

$$h(X) \equiv \limsup_{n \to \infty} \frac{\log |W_X(\Lambda_n^+)|}{n^d}$$

By subadditivity the limsup in the above definition is a limit and equals the infimum of all such values.

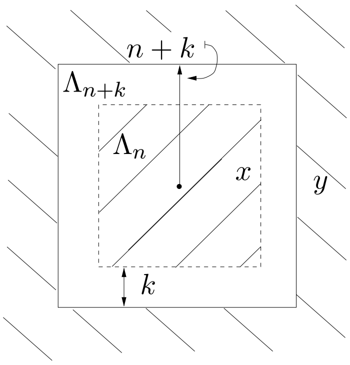


Figure 1: Square Mixing

Definition 2.2. A $\mathbb{Z}^d$ subshift $Y \subset S^{\mathbb{Z}^d}$ is a *shift of finite type* (SFT) if there exists an $n > 0$ such that

$$Y = \{x \in S^{\mathbb{Z}^d} : x|_{\Lambda_n + v} \in W_Y(\Lambda_n)\ \forall\ v \in \mathbb{Z}^d\}.$$

We say such an SFT is Λ_n *scaled.* In an analogous fashion SFTs may be $\mathcal{A}$ *scaled* for other subsets $\mathcal{A} \subset \mathbb{Z}^d$. If $Y \subset S^{\mathbb{Z}^d}$ is an $\mathcal{A}$ scaled SFT and $\mathcal{C} \subset \mathbb{Z}^d$, then a word $u \in S^{\mathcal{C}}$ is said to be $\mathcal{A}$ *locally* Y *allowed* if $\mathcal{A} + v \subset \mathcal{C}$ implies $u|_{\mathcal{A}+v} \in W_Y(\mathcal{A} + v)$.

We say $Y \subset S^{\mathbb{Z}^d}$ is a *matrix* SFT if there exists a set $\{A_i\}_{i=1}^d$ of $|S| \times |S|$, 0-1 matrices such that $Y = \{y \in S^{\mathbb{Z}^d} : A_i(y_v, y_{v+e_i}) = 1,\ \forall\ 1 \le i \le d\ \forall\ v \in \mathbb{Z}^d\}$ where for $1 \le i \le d$ the e_i are the standard unit vectors generating $\mathbb{Z}^d$.

Definition 2.3. A subshift X such that $x \neq \sigma^p x$ for every $x \in X$ and $p \in \mathbb{Z}^d \backslash 0$ is called *non-periodic* since it has no periodic points.

Definition 2.4. A $\mathbb{Z}^d$ subshift Y is *square mixing* (SM) if there exists k such that for each $n > 0$ and $x, y \in Y$ there exists $z \in Y$ such that $z|_{\Lambda_n} = x|_{\Lambda_n}$ and $z|_{\mathbb{Z}^d \backslash \Lambda_{n+k}} = y|_{\mathbb{Z}^d \backslash \Lambda_{n+k}}$.

The parameter k is referred to as the *square mixing gap parameter.* If a square mixing SFT contains a finite orbit, we call it a finite orbit square mixing SFT (FOSM SFT). The effect of square mixing upon the presence of finite orbits is not completely understood. It is not hard to see that, for $d = 2$, a $\mathbb{Z}^d$ square mixing SFT has finite orbits, as we show in Section 9 (see also [W]). Whether this must be the case for $d > 2$ is not known.

The following is our principal result.

Theorem 2.5. *For $d \geq 2$, let X be a non-periodic $\mathbb{Z}^d$ subshift, let Z be a $\mathbb{Z}^d$ finite orbit square mixing SFT and suppose there is a homomorphism $X \to Z$. Then, there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

We have two immediate corollaries. For Z to contain a finite orbit means there is a subgroup $G \subset \mathbb{Z}^d$ of finite index and a point in Z which is fixed by G. That is, for some $z \in Z$, $\sigma^g z = z$ for all $g \in G$.

Corollary 2.5.1. *Let X be a non-periodic $\mathbb{Z}^d$ subshift, let Z be a square mixing SFT, and let $G \subset \mathbb{Z}^d$ have finite index. If there exists a set of disjoint closed subsets X_{u+G} indexed by the elements of $\mathbb{Z}^2/G$ such that for $v \in \mathbb{Z}^d$, $\sigma^v X_{u+G} = X_{v+u+G}$ and if some $z \in Z$ is fixed by G, then there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

Proof. One can check that letting $\phi(x) = \sigma^v z$ if and only if $x \in X_{v+G}$ defines a homomorphism $X \to Z$. ■

Corollary 2.5.2. *For $d \geq 2$, let X be a non-periodic $\mathbb{Z}^d$ subshift and let Z be a $\mathbb{Z}^d$ square mixing SFT which contains a fixed point. Then, there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

In the sequel [L] we will prove the following.

Theorem 2.6. *If X is a non-periodic $\mathbb{Z}^2$ subshift and if Z is a $\mathbb{Z}^2$ square filling mixing SFT, then there exists a homomorphism $X \to Z$.*

Together, Theorems 2.6 and 2.5 imply the following extension of Krieger's Embedding Theorem.

Theorem 2.7. *Let X be a non-periodic $\mathbb{Z}^2$ subshift and let Z be a $\mathbb{Z}^2$ square filling mixing SFT. There exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

Interestingly, the question of whether square mixing SFTs must have finite orbits is highlighted here since Theorem 2.5 assumes the SM SFTs have finite orbits and we make use of these finite orbits in the proof. Whether one could prove Theorem 2.5 without making reference to finite orbits, is not clear.

We now briefly address the proof of Theorem 2.5 beginning with the necessity of the entropy condition. Because any square mixing SFT necessarily contains a periodic point ($\sigma^v x = x$ for some $v \in \mathbb{Z}^d \backslash 0$) any embedding of a non-periodic subshift into a square mixing SFT will necessarily be proper. Following A. Quas and P. Trow [QT] we say a subshift X is *entropy minimal* if for any subshift $X' \subsetneq X$ we have $h(X') < h(X)$. In Section 9 we prove that any SM SFT is entropy minimal. Thus any embedding of a non-periodic subshift into a SM SFT must have strictly less entropy.

We turn to the sufficiency of the entropy condition. In the sections that follow we proceed to construct an embedding $\psi : X \hookrightarrow Z$. The assumption that we have a homomorphism $\phi : X \to Z$ in Theorem 2.5 permits a construction of an embedding which is very similar in spirit and shares analogous structures with the embedding constructed for $\mathbb{Z}$ subshifts (Krieger's Embedding Theorem). The principal difference is the geometry of $\mathbb{R}^d$; "Voronoi" tilings of $\mathbb{R}^d$ are much more complex for $d \geq 2$ than for $d = 1$. This construction takes some time, so we give an overview of it.

In Section 3 we introduce $\mathfrak{M}_m$, the class of m regular subsets of $\mathbb{Z}^d$ ($\mathcal{M} \subset \mathbb{Z}^d$ is m regular if no two elements of $\mathcal{M}$ are within m of each other and no element of $\mathbb{Z}^d$ is more than m from the set $\mathcal{M}$). Then we define and derive Voronoi tilings from m regular subsets of $\mathbb{Z}^d$. Such tilings consist of a collection of convex polyhedrons such that 1) for each polyhedron $\mathcal{V}$, $\overline{B}(p, m/2) \subset \mathcal{V} \subset \overline{B}(q, 2m)$ for some $p, q \in \mathbb{R}^d$ and 2) they cover $\mathbb{R}^d$ regularly (*i.e.*, distinct polyhedrons have disjoint interiors).

In Section 4 we associate the non-periodic subshift X to a set of Voronoi tilings. Namely, we construct a clopen set $F \subset X$ such that for each $x \in X$, the set $\mathcal{M}_x \equiv \{v \in \mathbb{Z}^d : \sigma^v x \in F\}$ is m regular. Thus, using the techniques of Section 3 we can derive Voronoi tilings from the sets $\mathcal{M}_x \in \mathfrak{M}_m$.

In Sections 5 and 6 we locate a Z allowed word W_1 which can act as a "marker" in the image. Specifically, in Section 5 we show there exist M_0 and a word $W_0 \in W_Z(\Lambda_{M_0})$ such that the subshift $Y \equiv \{z \in Z : z|_{\Lambda_{M_0}+v} \neq W_0 \ \forall\ v \in \mathbb{Z}^d\}$ is a SM SFT which contains a finite orbit, $h(X) < h(Y)$, and there exists a homomorphism $X \to Y$. In Section 6 we show there exist M', M_1 and a word $W_1 \in W_Z(\Lambda_{M_1}) \backslash W_Y(\Lambda_{M_1})$ such that for m suitably large, any $y \in Y$ and $\mathcal{M} \in \mathfrak{M}_m$, there exists $z \in Z$ such that $z_v = W_1 \iff v \in \mathcal{M}$ and $z = y$ on $\mathbb{Z}^d \backslash \cup_{v \in \mathcal{M}} \Lambda_{M'}$.

In Section 7 we construct an injection of words. Roughly speaking, for m suitably large and any convex polyhedron $\mathcal{V}$ such that $\overline{B}(p, m/2) \subset \mathcal{V} \subset \overline{B}(q, 2m)$, $p, q \in \mathbb{R}^d$, we construct an injection Ψ from $W_X(\mathcal{V})$ to Y allowed words which have been "marked" by W_1 and which have a fixed pattern of symbols on their boundary. Finally, in Section 8 (for each $x \in X$) we use the homomorphism to determine a pattern of symbols on the boundary of each Voronoi tile in the Voronoi tiling constructed from $\mathcal{M}_x$ and we use the injection Ψ to paint "marked" Y allowed words on the interiors of each Voronoi tile producing an element $\psi(x) \in Z$. The results from the previous sections are then applied to show $\psi : X \to Z$ is an embedding and that completes the proof of Theorem 2.5.

3. Voronoi Tilings.

In this section we introduce a couple fairly standard Voronoi tiling definitions (see [A] [OBS]) and prove a simple lemma (3.3). We then define regular coverings and prove a result (Lemma 3.5) summarizing what we need to know about such coverings.

Definition 3.1. A set $\mathcal{M} \subset \mathbb{R}^d$ is m *separated* if for every $v \in \mathcal{M}$, $\overline{B}(v, m) \cap \mathcal{M} = \{v\}$. A set $\mathcal{M} \subset \mathbb{R}^d$ is m *syndetic with respect to* $\mathbb{Z}^d$ if $v \in \mathbb{Z}^d \ \Rightarrow\ \overline{B}(v, m) \cap \mathcal{M} \neq \emptyset$. A m separated set $\mathcal{M} \subset \mathbb{R}^d$ which is m syndetic with respect to $\mathbb{Z}^d$ will be called m *regular*. We define $\mathfrak{M}_m = \{\mathcal{M} \subset \mathbb{Z}^d : \mathcal{M} \text{ is } m \text{ regular }\}$.

If $\mathcal{M} \subset \mathbb{Z}^d$ is m syndetic with respect to $\mathbb{Z}^d$, then $\mathcal{M} \cap \overline{B}(v, m + \sqrt{d}/2) \neq \emptyset$, for all $v \in \mathbb{R}^d$.

Definition 3.2. For $v \in \mathcal{M} \subset \mathbb{R}^d$ the *Voronoi tile* corresponding to v with respect to $\mathcal{M}$ is the set $\mathcal{V}_v \equiv \{r \in \mathbb{R}^d : \rho(r,v) \leq \rho(r,\mathcal{M})\}$ (where ρ is the Euclidean metric).

The following is well known but we prove it anyway.

Lemma 3.3. *If $\mathcal{M} \subset \mathbb{R}^d$ is m regular, then for each $v \in \mathcal{M}$ the Voronoi prototile $\mathcal{V}_v$ is a closed convex polyhedron and $\overline{B}(v, m/2) \subset \mathcal{V}_v \subset \overline{B}(v, m+\sqrt{d}/2)$.*

Proof. Let $x \in \mathcal{V}_v$ and suppose $\rho(x,v) > m + \sqrt{d}/2$. Since $\mathcal{V}_v$ is the Voronoi tile corresponding to v with respect to $\mathcal{M}$, and $x \in \mathcal{V}_v$, it follows that x is as close to v as to any other element of $\mathcal{M}$, that is $\rho(x,\mathcal{M}) > m + \sqrt{d}/2$. But this means $\mathcal{M} \cap \overline{B}(x, m + \sqrt{d}/2) = \emptyset$ contradicting $\mathcal{M}$'s m syndeticity with respect to $\mathbb{Z}^d$. Thus the Voronoi tile $\mathcal{V}_v \subset \overline{B}(v, m+\sqrt{d}/2)$.

On the other hand, if there exists a point $x \notin \mathcal{V}_v$ and $d(v,x) \leq m/2$ then there exists another point $u \in \mathcal{M}$ such that $d(u,x) < m/2$. But then $d(v,u) \leq d(v,x) + d(x,u) < m/2 + m/2 = m$, a contradiction since $d(v,u) > m$ for each distinct pair in $\mathcal{M}$. Thus $\overline{B}(v, m/2) \subset \mathcal{V}_v$.

Each Voronoi tile $\mathcal{V}_v = \bigcap_{u \in \mathcal{M}_v} H_v(u)$ where $\mathcal{M}_v = \mathcal{M} \backslash \{v\}$ and $H_v(u) = \{x \in \mathbb{R}^d : \|x - v\| \leq \|x - u\|\}$ is the closed halfspace containing v. Thus the Voronoi tile $\mathcal{V}_v$ is the intersection of a collection of closed convex objects and therefore is closed and convex. Since $\mathcal{V}_v \subset \overline{B}(v, m + \sqrt{d}/2)$ and since $\mathcal{V}_v = \cap H_v(u)$ it suffices to take the intersection over the finite set of points $u \in \overline{B}(v, 2m + \sqrt{d}) \cap \mathcal{M}_v$. Therefore, the Voronoi tile $\mathcal{V}_v$ is a closed convex polyhedron. ∎

Corollary 3.3.1. *The set $\mathfrak{F}_m \equiv \{\mathcal{V}_v - v : v \in \mathcal{M}$ for some $\mathcal{M} \in \mathfrak{M}_m\}$ is finite.*

Proof. For $\mathcal{M} \in \mathfrak{M}_m$ and $v \in \mathcal{M}$ each $\mathcal{V}_v \subset \overline{B}(v, m + \sqrt{d}/2)$ and this means that each

$$\mathcal{V}_v - v = \{r \in \mathbb{R}^d : \rho(r, 0) \leq \rho(r, \mathcal{M}' \cap \overline{B}(0, 2m + \sqrt{d}))\}$$

where $\mathcal{M}' = \mathcal{M} - v \subset \mathbb{Z}^d$ is m regular and contains the origin. Because the collection of all possible m separated subsets of $\overline{B}(0, 2m + \sqrt{d}) \cap \mathbb{Z}^d$ is finite so is the set of all such $\mathcal{V}_v - v$. ∎

The set $\{\mathcal{V}_v\}_{v\in\mathcal{M}}$ of Voronoi tiles is a cover of $\mathbb{R}^d$ and for $u \neq v$ the sets $\mathcal{V}_u$ and $\mathcal{V}_v$ have disjoint interiors, however the collection of sets $\{\mathcal{V}_v\}_{v\in\mathcal{M}}$ is not necessarily uniquely associated to a subset $\mathcal{M} \in \mathfrak{M}_m$. There exist distinct $\mathcal{M}, \mathcal{M}' \in \mathfrak{M}_m$ for which the collections $\{\mathcal{V}_v\}_{v\in\mathcal{M}}$ and $\{\mathcal{V}'_{v'}\}_{v'\in\mathcal{M}'}$ are the same, where $\mathcal{V}'_{v'}$ is the Voronoi tile corresponding to v' with respect to $\mathcal{M}'$. In fact, given $\mathcal{M}$, the collection of such $\mathcal{M}'$ (with identical Voronoi tilings) may be uncountable. For this reason, we take the following approach to tiling.

Let $\mathfrak{F}$ be a finite collection of convex polyhedral sets and suppose there exists a set $\mathfrak{V} \subset \mathfrak{F} \times \mathbb{Z}^d$ such that $\mathbb{R}^d = \bigcup_{(\mathcal{V},z)\in\mathfrak{V}} \mathcal{V} + z$ and distinct $(\mathcal{V}, z), (\mathcal{V}', z') \in \mathfrak{V}$ are such that $(\mathcal{V} + z) \cap (\mathcal{V}' + z')$ has empty interior. Then $\mathfrak{V}$ is referred to as a *regular* $\mathfrak{F}$ *covering* of $\mathbb{R}^d$ and $\mathfrak{F}$ is referred to as the set of *prototiles.* Regular $\mathfrak{F}$ coverings of $\mathbb{R}^d$ embody our notion of a tiling of $\mathbb{R}^d$ by polyhedral sets.

Let us define a shift σ on the elements of $\mathfrak{F} \times \mathbb{Z}^d$. For $\beta = (\mathcal{V}, u) \in \mathfrak{F} \times \mathbb{Z}^d$ and $v \in \mathbb{Z}^d$, define $\sigma^v(\beta) = (\mathcal{V}, u - v)$. For $\mathfrak{V} \subset \mathfrak{F} \times \mathbb{Z}^d$ let us write $\sigma^v(\mathfrak{V})$ for the set $\{\sigma^v(\beta) : \beta \in \mathfrak{V}\}$. Clearly, $\mathfrak{V}$ is a regular $\mathfrak{F}$ covering of $\mathbb{R}^d$ if and only if $\sigma^v(\mathfrak{V})$ is.

We would like to define a topology on collections of regular $\mathfrak{F}$ coverings in which whenever two regular $\mathfrak{F}$ coverings $\mathfrak{V}_1$ and $\mathfrak{V}_2$ agree on some finite set $\mathfrak{F} \times \Lambda_n$, then they lie in a common basis set. Such collections of regular $\mathfrak{F}$ coverings are subsets of $\mathcal{P}(\mathfrak{F} \times \mathbb{Z}^d)$, where $\mathcal{P}(\mathcal{A})$ denotes the collection of all subsets of $\mathcal{A}$. Also, since $\mathfrak{F}$ is finite, $\mathfrak{F} \times \mathbb{Z}^d$ is a discrete metric space.

Definition 3.4. For a discrete metric space A define the *Bounded Neighborhood* (BN) topology on $\mathcal{P}(A)$ as that topology generated by the basis of sets consisting of all $\mathcal{U}(\alpha, B) \equiv \{\beta \in \mathcal{P}(A) : \alpha \cap B = \beta \cap B\}$ where $B \subset A$ is finite and $\alpha \in \mathcal{P}(A)$

A collection V of regular $\mathfrak{F}$ coverings of $\mathbb{R}^d$ may be given the (BN) topology since $V \subset \mathcal{P}(\mathfrak{F} \times \mathbb{Z}^d)$. With respect to this topology, the shift map σ defined above is continuous. The following lemma summarizes what we need to know about Voronoi tilings generated by sets $\mathcal{M} \in \mathfrak{M}_m$.

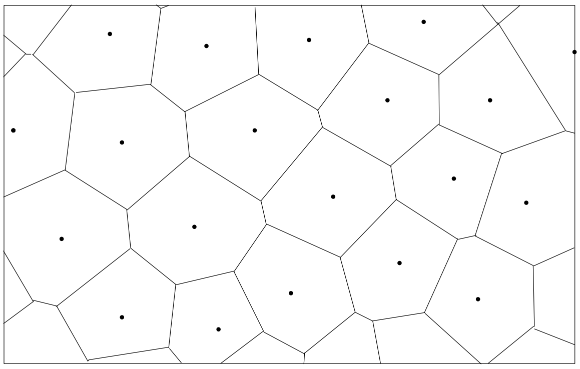


Figure 2: $\mathcal{M}$ and the Corresponding Voronoi Tiling $\mathfrak{V}_m(\mathcal{M})$.

Lemma 3.5. *For $m \geq 1$ fixed, the collection $\mathfrak{F}_m \equiv \{\mathcal{V}_v - v : v \in \mathcal{M}$ for some $\mathcal{M} \in \mathfrak{M}_m\}$ is a finite set of convex polyhedral prototiles such that for each prototile $\mathcal{V} \in \mathfrak{F}_m$, $\overline{B}(0, m/2) \subset \mathcal{V} \subset \overline{B}(0, m + \sqrt{d}/2)$. For any $\mathcal{M} \in \mathfrak{M}_m$, the set $\mathfrak{V}_m(\mathcal{M}) \equiv \{(\mathcal{V}_v - v, v) : v \in \mathcal{M}\}$ is a well-defined regular $\mathfrak{F}_m$ covering of $\mathbb{R}^d$ with the following properties.*

1) For every $v \in \mathcal{M}$ there exists a unique $(\mathcal{V}, u) \in \mathfrak{V}_m(\mathcal{M})$ such that $u = v$ and $\mathcal{V} + u$ is the Voronoi prototile corresponding to v with respect to $\mathcal{M}$.

2) The map $\mathcal{M} \mapsto \mathfrak{V}_m(\mathcal{M})$ is (BN)-continuous.

3) $\mathfrak{V}_m(\sigma^v \mathcal{M}) = \sigma^v \mathfrak{V}_m(\mathcal{M})$.

We refer to the set $\mathfrak{F}_m$ as the set of *Voronoi prototiles* and to $\mathfrak{V}_m(\mathcal{M})$ as the *Voronoi tiling of $\mathbb{R}^d$ generated by $\mathcal{M}$*. See Figure 2 for a picture of an example for $d = 2$. Note $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}) \iff v \in \mathcal{M}$.

Proof of 3.5. The first sentence follows from Lemma 3.3 and Corollary 3.3.1. Since $\mathfrak{V}_m(\mathcal{M})$ is the collection of pairs $(\mathcal{V}, v) \in \mathfrak{F}_m \times \mathbb{Z}^d$ such that *i)* $v \in \mathcal{M}$ and *ii)* $\mathcal{V} \in \mathfrak{F}_m$ is the unique element such that $\mathcal{V} + v = \mathcal{V}_v$, it follows that for $v \in \mathcal{M}$ there exists only one element $(\mathcal{V}, u) \in \mathfrak{V}_m(\mathcal{M})$ such that $u = v$ (and $\mathcal{V}_v = \mathcal{V} + u$).

The continuity of the map $\mathcal{M} \mapsto \mathfrak{V}_m(\mathcal{M})$ may be demonstrated as follows. For $k > 0$ let $K = k + 2m + \sqrt{d}$. For $\mathcal{M}, \mathcal{M}' \in \mathfrak{M}_m$ let $\{\mathcal{V}_v\}_{v \in \mathcal{M}}$ and $\{\mathcal{V}'_v\}_{v \in \mathcal{M}'}$ denote the Voronoi tiles corresponding to the elements of $\mathcal{M}$ and $\mathcal{M}'$, respectively. For each $v \in \mathcal{M}$ the Voronoi tile $\mathcal{V}_v$ is determined by $\mathcal{M} \cap \overline{B}(v, 2m + \sqrt{d})$ (likewise for $\mathcal{M}'$). If $\mathcal{M}$ and $\mathcal{M}'$ agree on Λ_K, then for every $v \in \mathcal{M}|_{\Lambda_k} = \mathcal{M}'|_{\Lambda_k}$ (since $\overline{B}(v, 2m + \sqrt{d}) \subset \Lambda_K$) we have $\mathcal{V}_v = \mathcal{V}'_v$. Thus $\mathfrak{V}_m(\mathcal{M})$ and $\mathfrak{V}_m(\mathcal{M}')$ agree on $\mathfrak{F}_m \times \Lambda_k$.

Part 3 is just a matter of observing that shifting a set $\mathcal{M}$, shifts its Voronoi tiling and therefore shifts the corresponding $\mathfrak{V}_m(\mathcal{M})$. ■

4. Markers in the Domain.

We now associate Voronoi tilings to a non-periodic subshift X. We do this by constructing a continuous shift commuting map $X \to \mathfrak{M}_m$ for suitable m $(x \mapsto \mathcal{M}_x)$. For each x, the image $\mathcal{M}_x$ of "markers" is derived from a clopen "marking set" and is viewed as presenting local periodicity information about the symbols that occur in x. This is done much as is done in the proof of Krieger's embedding theorem for $\mathbb{Z}$ subshifts. We proceed as follows.

For $\mathcal{A} \subset \mathbb{R}^d$ and $j \in \mathbb{Z}^d \backslash \{0\}$, a word $w \in W_X(\mathcal{A})$ said to be j *periodic* if for every pair $v, v + j \in \mathcal{A} \cap \mathbb{Z}^d$ we have $w_v = w_{v+j}$. For a $\mathbb{Z}^d$ subshift X and positive integers m and M with $0 < m < M$, let $\{w_i\}_{i=1}^K$ be an enumeration of the following set of words.

$$\{w_i\}_{i=1}^K \equiv \big\{w \in W_X(\Lambda_M) : w \text{ is not } j \text{ periodic for any } j \in \overline{B}(0, m)\backslash\{0\}\big\}. \qquad (1)$$

Let $(w_i) \equiv \{x \in X : x|_{\Lambda_M} = w_i\}$; so (w_i) is a clopen subset of X. We have the following algorithm.

Algorithm 4.1. For integers $m < M$ and an enumeration of words $\{w_i\}_{i=1}^K$, we construct the set $F(m, M) \subset X$ as follows. Let $F_1 = (w_1)$, for $1 < n \leq K$ let $F_n = F_{n-1} \cup ((w_n)\backslash \bigcup_{j \in \overline{B}(0,m)} \sigma^j F_{n-1})$, and let $F(m, M) = F_K$. •

Reindexing the words $\{w_i\}_{i=1}^K$ may produce a different set $F(m, M)$, however any such $F(m, M)$ will be adequate for us.

Lemma 4.2. *For positive integers m and M with $0 < m < M$ and $\{w_i\}_{i=1}^K$ as in Equation 1, the set $F = F(m, M)$ is a clopen subset of X with the following properties:*

1) $F \cap \sigma^j F = \emptyset$, $\forall\, j \in \overline{B}(0, m)\backslash\{0\}$

2) $\bigcup_{i=1}^{K} (w_i) \subset \bigcup_{j \in \overline{B}(0,m)} \sigma^j F$.

Proof. The (w_i) have the property that $(w_i) \cap \sigma^j(w_i) = \emptyset, \ \forall \ j \in \overline{B}(0,m)\backslash\{0\}$. The definition of F_n preserves clopenness, and property 1; that is, if F_{n-1} and (w_n) are clopen so is F_n, and if F_{n-1} and (w_n) satisfy property 1 so does F_n. Lastly $\bigcup_{j\in\overline{B}(0,m)} \sigma^j F_{n-1} \cup (w_n) \subset \bigcup_{j\in\overline{B}(0,m)} \sigma^j F_n$ which applied repeatedly beginning with $F = F_K$ gives us property 2. ∎

Corollary 4.2.1. *For m, M, and F as in Lemma 4.2 and $x \in X$, if $x \notin \bigcup_{j\in\overline{B}(0,m)} \sigma^j F$, then $x|_{\Lambda_M}$ is a j periodic word for some $j \in \overline{B}(0,m)\backslash\{0\}$.*

We fix $F = F(m,M)$ and use it to construct for each $x \in X$ a set of *markers*

$$\mathcal{M}_x \equiv \{v \in \mathbb{Z}^d : \sigma^v x \in F\}. \tag{2}$$

Corollary 4.2.2. *$\mathcal{M}_x$ has the following properties (for $v \in \mathbb{Z}^d$):*

1) if $v \in \mathcal{M}_x$, then $\overline{B}(v,m) \cap \mathcal{M}_x = \{v\}$ ($\mathcal{M}_x$ is m separated) and

2) if $\overline{B}(v,m) \cap \mathcal{M}_x = \emptyset$ then $x|_{\Lambda_M+v}$ is j periodic for some $j \in \overline{B}(0,m)\backslash\{0\}$.

Remark 4.3. The map $x \mapsto \mathcal{M}_x$ for $x \in X$ is shift commuting and continuous in the sense that given $k > 0$ there exists K such that whenever $x|_{\Lambda_K} = y|_{\Lambda_K}$ then $\mathcal{M}_x|_{\Lambda_k} = \mathcal{M}_y|_{\Lambda_k}$ and $\sigma^v\mathcal{M}_x = \mathcal{M}_{\sigma^v x}$, where $\sigma^v\mathcal{M}_x = \mathcal{M}_x - v$.

Lemma 4.4. *Given a non-periodic subshift X and an integer $m > 0$ there exists an integer $M > m$ such that for any $F = F(m,M)$ and each $x \in X$, the set $\mathcal{M}_x \in \mathfrak{M}_m$.*

Proof. That $\mathcal{M}_x$ is m separated follows from Corollary 4.2.2 Part 1. That $\mathcal{M}_x$ is m syndetic with respect to $\mathbb{Z}^d$ follows because X is compact. Specifically, we claim there exists M such that for every $x \in X$, the word $x|_{\Lambda_M}$ is not p periodic for any $p \in \overline{B}(0,m)\backslash\{0\}$. Suppose not, then there exists a sequence $M_k \to \infty$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k|_{\Lambda_{M_k}}$ p periodic for some $p \in \overline{B}(0,m)\backslash\{0\}$. It follows that there is a subsequence $x_{k'}$ which converges to some point $x \in X$ and such that $x_{k'}|_{\Lambda_{M_{k'}}}$ are all p periodic for some $p \in \overline{B}(0,m)\backslash\{0\}$. One can check that x is p periodic, contradicting the non-periodicity of X.

As demonstrated in Corollary 4.2.2 Part 2, it is the nature of $F = F(m, M)$ and hence the derived set $\mathcal{M}_x$ that if $(\overline{B}(0, m) + v) \cap \mathcal{M}_x = \emptyset$ for some $v \in \mathbb{Z}^d$, then $x|_{\Lambda_M + v}$ is p periodic for some $p \in \overline{B}(0, m)\backslash\{0\}$, thus $(\overline{B}(0, m) + v) \cap \mathcal{M}_x \neq \emptyset$. ■

5. The Abundance of FOSM SFTs.

In this section we develop an aspect of finite orbit square mixing (FOSM) SFTs which is needed to prove Theorem 2.5. Specifically, we show that any FOSM SFT Z contains an abundance of subshifts which are themselves FOSM SFTs and which may have entropy arbitrarily close to $h(Z)$. In what follows we refer to an SFT which contains more than a single point as a *nontrivial* SFT.

Theorem 5.1. *If Z is a nontrivial $\mathbb{Z}^d$ FOSM SFT, then for any $\epsilon > 0$ there exist $M_0 > 0$ and a word $W_0 \in W_Z(\Lambda_{M_0})$ such that $Y \equiv \{z \in Z : z|_{v+\Lambda_{M_0}} \neq W_0, \ \forall\ v \in \mathbb{Z}^d\}$ is a FOSM SFT with $Y \subsetneq Z$ and $h(Y) > h(Z) - \epsilon$. Moreover, if Y' is a subshift with $Y' \subsetneq Z$, then W_0 may be chosen such that $Y' \subset Y$.*

Remark 5.2. That Z has a point with a finite orbit is crucial for us since $W_0 = x|_{\Lambda_{M_0}}$ for some $x \in Z$ with a finite orbit. Whether this is necessary is not clear.

Remark 5.3. For $d = 2$, Theorem 5.1 also holds if we replace FOSM SFT with square filling mixing SFT.

Lemma 5.4. *In a FOSM SFT the set of points whose orbits are finite is a dense set.*

The proof of this lemma follows naturally from the arguments used to prove the entropy relation in Theorem 5.1 so we will not present a separate proof. Rather, we will address the proof by merely making a remark at the appropriate point in the proof of Theorem 5.1.

The rest of the section will be directed toward the proof of Theorem 5.1. We begin by setting down some notation and a couple facts. For $n > 1$ and $m \geq 3$ let

$\Psi_{m,n} \equiv \Lambda^+_{(m-2)n} + (n, .., n)$ and

$A_{m,n} \equiv \Lambda_{mn}^{+} \backslash \Psi_{m,n}$.

Then Λ_{mn}^{+} is the disjoint union of the annulus $A_{m,n}$ and the shifted hypercube $\Psi_{m,n}$.

For $v_1, .., v_d \in \mathbb{Z}^d$ let $\langle v_1, .., v_d \rangle$ denote the subgroup they generate. For $1 \leq j \leq d$ let e_j denote the j-th canonical basis element $(0, .., 1, .., 0) \in \mathbb{Z}^d$ whose j-th component is 1 and the other components are zero. For $n \in \mathbb{Z}$ and $t = (t_1, .., t_d) \in \mathbb{Z}^d$ let $nt = (nt_1, .., nt_d)$. If $G \subset \mathbb{Z}^d$ is a subgroup with $|\mathbb{Z}^d/G| < \infty$, then for each $1 \leq j \leq d$ there exists an $n_j \in \mathbb{N}$, $n_j > 0$ such that $n_j e_j \in G$ (hence, there exists an $n \in \mathbb{N}$, $n > 0$ such that $ne_j \in G$ for $1 \leq j \leq d$).

For $w \in \mathbb{Z}^d$ and $K, m, n, n_1, .., n_d \in \mathbb{N}$ such that $m \geq 3$, $n_i > 0$, $n > \max\{n_1, .., n_d\}$ and $K \geq mn$, if $\Psi_{m,n} \cap (\Lambda_K^{+} + w) \neq \emptyset$ then for any $a \in \mathbb{Z}^d$ there exist $k_i \in \mathbb{Z}$ such that $a + \sum_{i=1}^{d} k_i v_i \in A_{m,n} \cap (\Lambda_K^{+} + w)$ where $v_i = n_i e_i$.

Before we begin the proof of Theorem 5.1 we present a preliminary "replacement" result. Let Z be a nontrivial matrix SFT and suppose $x \in Z$ has a finite orbit. As such there exist $v_1, .., v_d \in \mathbb{Z}^d$ such that $|\mathbb{Z}^d/\langle v_1, .., v_d \rangle| < \infty$ and $\sigma^{v_i} x = x$ for all $i = 1, .., d$. Since we may, let us assume each $v_i = n_i e_i$ for some $n_i > 0$. Let m, n be such that $m \geq 3$, $n > \max\{n_1, .., n_d\}$ and let $V \in W_Z(\Lambda_{mn}^{+})$ be any Z allowed word such that $V|_{A_{m,n}} = x|_{A_{m,n}}$ but $V \neq x|_{\Lambda_{mn}^{+}}$. (We will not concern ourselves with V's existence at this point since that will be guaranteed in due time.) If $y \in Z$ is such that $y|_{\Lambda_{mn}^{+}} = x|_{\Lambda_{mn}^{+}}$, then because Z is a matrix SFT we may "construct" or define a new point $\widehat{y} \in Z$ such that $\widehat{y}|_{\mathbb{Z}^d \backslash \Lambda_{mn}^{+}} = y|_{\mathbb{Z}^d \backslash \Lambda_{mn}^{+}}$ and $\widehat{y}|_{\Lambda_{mn}^{+}} = V$.

Let $R \subset \mathbb{Z}^d$ be any sufficiently large square, that is $R = \Lambda_K^{+} + v'$ for some $v' \in \mathbb{Z}^d$ and $K \in \mathbb{N}$, where $K \geq mn$. For $y \in Z$ define

$$B(y) = \{w \in \mathbb{Z}^d : \sigma^w y|_R = x|_R\}.$$

Lemma 5.5. *For $y \in Z$ such that $y|_{\Lambda_{mn}^{+}} = x|_{\Lambda_{mn}^{+}}$ and for $\widehat{y}$ as defined above, $B(\widehat{y}) \subsetneq B(y)$.*

If we think of the construction of $\widehat{y}$ from y as a replacement procedure or algorithm that replaces the word $x|_{\Lambda_{mn}^{+}}$ (which looks like x) with the word V (which doesn't look like x), then Lemma 5.5 tells us that this replacement procedure doesn't create any new words in $\widehat{y}$ which look like x.

Proof of 5.5. Writing $z = \widehat{y}$, we suppose $w \in B(z)$, that is $\sigma^w z|_R = x|_R$. This is equivalent to saying $z_a = x_{a-w}$ for all $a \in R + w$. If $\Psi_{m,n} \cap (R + w) = \emptyset$, then for all $a \in R + w$, $y_a = z_a = x_{a-w}$ which implies $\sigma^w y|_R = x|_R$. That is, $w \in B(y)$.

So suppose w is such that $\Psi_{m,n} \cap (R+w) \neq \emptyset$ and suppose $a \in R+w$ (hence $z_a = x_{a-w}$). There exist $k_1, .., k_d \in \mathbb{Z}$ such that

$$a + \sum k_i v_i \in A_{m,n} \cap (R + w).$$

The periodicity of x implies $x_{a-w} = x_{a-w+\sum k_i v_i}$. Since $a + \sum k_i v_i \in R + w$ this implies $z_{a+\sum k_i v_i} = x_{a+\sum k_i v_i - w}$ and thus (since $a \in R + w$)

$$z_a = x_{a-w} = x_{a+\sum k_i v_i - w} = z_{a+\sum k_i v_i}.$$

Now, either $a \in \Psi_{m,n}$ or not. If so, then since $a, a + \sum k_i v_i \in \Lambda^+_{mn}$, and $y|_{\Lambda^+_{mn}} = x|_{\Lambda^+_{mn}}$ we know $y_a = y_{a+\sum k_i v_i}$. And since $a + \sum k_i v_i \in A_{m,n}$, $y_{a+\sum k_i v_i} = z_{a+\sum k_i v_i}$. Thus

$$z_a = z_{a+\sum k_i v_i} = y_{a+\sum k_i v_i} = y_a.$$

On the other hand, if a is not in $\Psi_{m,n}$ then $y_a = z_a$. In either case, $y_a = z_a$ for any $a \in R + w$. Thus $\sigma^w y|_R = \sigma^w z|_R = x|_R$, so again $w \in B(y)$. In sum $B(z) \subset B(y)$. ∎

Proof of 5.1. We may assume Z is a square mixing *matrix* SFT with square mixing gap parameter k. By assumption there exists an $x \in Z$ and $v_1, .., v_d \in \mathbb{Z}^d$ such that $|\mathbb{Z}^d / \langle v_1, .., v_d \rangle| < \infty$ and $\sigma^{v_i} x = x$ for each $i = 1, .., d$. We assume each $v_i = n e_i$ where $n > k$. We let $\alpha = x|_{\Lambda^+_n}$. The periodicity of x implies $\alpha = x|_{\Lambda^+_n + nt}$ for all $t \in \mathbb{Z}^d$ where $nt = (nt_1, .., nt_d)$. That is, x may be regarded as formed by the concatenation of the word α. Similarly (for $m \geq 3$) the word $\partial V = x|_{A_{m,n}}$ is the concatenation of the word α around the annulus $A_{m,n}$.

Because we have assumed Z is nontrivial, we know for m suitably large that there exists a word $B \in W_Z(\Lambda^+_{(m-4)n})$ such that $B \neq x|_{\Lambda^+_{(m-4)n}}$. Since Z is square mixing and $n > k$, there exists a word $V \in W_Z(\Lambda^+_{mn})$ such that $V|_{A_{m,n}} = \partial V$ and $V|_{\Lambda^+_{(m-2)n} + ne} \neq x|_{\Lambda^+_{(m-2)n}}$ where $e = (1, .., 1)$. To see this, let $y \in Z$ be such that $y|_{\Lambda^+_{(m-4)n} + 2ne} = B$. By square mixing, there exists $z \in Z$ such that $z|_{\Lambda^+_{(m-4)n}+2ne} = y|_{\Lambda^+_{(m-4)n}+2ne}$ and $z|_{\mathbb{Z}^2\setminus\Psi_{m,n}} = x|_{\mathbb{Z}^2\setminus\Psi_{m,n}}$. So, $V = z|_{\Lambda^+_{mn}}$.

Remark 5.6. The existence of x and V allows us to prove Lemma 5.4. As described above, for any m suitably large and any $B \in W_Z(\Lambda^+_{(m-4)n})$ there exists a $V_B \in W_Z(\Lambda^+_{mn})$ such that $V_B|_{\Lambda^+_{(m-4)n}+2ne} = B$, and $V_B|_{A_{m,n}} = \partial V$. Thus, the point z defined by $z|_{\Lambda^+_{mn}+mnt} = V_B$ for all $t \in \mathbb{Z}^d$ is an element of Z and $\sigma^v z = z$ whenever $v = mnt$, $t \in \mathbb{Z}^d$. Since $z|_{\Lambda^+_{(m-4)n}+2ne} = B$ and z has a finite orbit, we see the set of points in Z whose orbits are finite is dense, proving Lemma 5.4.

We now show the following claim.

Claim 1. *For m suitably large if we let $M_0 = mn$ and $W_0 = x|_{\Lambda_{mn}}$, then $h(Y) > h(Z) - \epsilon$.*

Proof. From the definition of entropy, given ϵ there exists m such that

$$|W_Z(\Lambda^+_{((m-4)n)})| - 1 > e^{(h(Z)-\epsilon)m^d n^d}. \tag{3}$$

As described earlier, for each word $\Delta \in W_Z(\Lambda^+_{(m-4)n})$ there exists (by square mixing) at least one word $U_\Delta \in W_Z(\Lambda^+_{mn})$ such that $U_\Delta|_{\Lambda^+_{(m-4)n}+2ne} = \Delta$ and $U_\Delta|_{A_{m,n}} = \partial V$. Let $N = |W_Z(\Lambda^+_{((m-4)n)})| - 1$ and let $\{\Delta_i\}_{i=1}^N = W_Z(\Lambda^+_{(m-4)n})\setminus\{x|_{\Lambda^+_{(m-4)n}}\}$. Since Z is a matrix SFT, for each $\eta \in N^{\mathbb{Z}^d}$ there exists $z = z(\eta)$ such that $z|_{\Lambda^+_{mn}+mnt} = U_{\Delta_{\eta_t}}$ for $t \in \mathbb{Z}^d$. For points z so constructed, the number of words $z|_{\Lambda^+_{kmn}}$ would equal $N^{(k^d)}$. (Also, a countable number of such points z will have a finite orbit.)

Note, if Y were defined using $x|_{\Lambda^+_{mn}}$ instead of W_0, then there could exist elements z (as constructed above) which need not be in Y. Since Y is defined using W_0, let us check that the points z are elements of Y. We will do this by contradiction; suppose $\sigma^w z|_{\Lambda_{mn}} = x|_{\Lambda_{mn}}$ or equivalently $z|_{\Lambda_{mn}+w} = x|_{\Lambda_{mn}}$. The periodicity of x implies for every $t \in \mathbb{Z}^d$ such that $a, a+nt \in \Lambda_{mn} + w$ that

$$z_a = z_{a+nt}. \tag{4}$$

For $t' \in \mathbb{Z}^d$, let $\kappa = mnt'$ be such that $\Lambda_{mn} + w \supset \Lambda^+_{mn} + \kappa$. By the construction of z, there exists $\Delta \in W_Z(\Lambda^+_{(m-4)n})$ such that for any $a \in \Lambda^+_{(m-4)n} + \kappa + 2ne$,

$$z_a = \Delta_{a-\kappa-2ne}. \tag{5}$$

Since $a \in \mathbb{Z}^d$, there exist $t \in \mathbb{Z}^d$ such that $a+nt \in A_{m,n}+\kappa$. On the other hand, $a' \in A_{m,n}+\kappa$ implies $z_{a'} = x_{a'-\kappa}$. Together with Equations 4 and 5 this means (with $a' = a + nt$)

$$\Delta_{a-\kappa-2ne} = z_a = z_{a+nt} = x_{a+nt-\kappa}$$

for all $a \in \Lambda^+_{(m-4)n} + \kappa + 2ne$. That is, for all $a'' \in \Lambda^+_{(m-4)n}$, $\Delta_{a''} = x_{a''+2ne+nt}$ and thus $\Delta_{a''} = x_{a''}$ because of the periodicity of x. This implies $\Delta = x|_{\Lambda^+_{(m-4)n}}$, contradicting our choice of Δ. In conclusion $z|_{\Lambda_{mn}+w} \neq x|_{\Lambda_{mn}}$ for any $w \in \mathbb{Z}^d$ and z so constructed. So $z \in Y$.

We can underestimate the number of words in $W_Y(\Lambda^+_{kmn})$ by counting only those words $z|_{\Lambda^+_{kmn}}$. So, for any $k \in \mathbb{N}, k \geq 1$ we have $|W_Y(\Lambda^+_{kmn})| \geq N^{(k^d)} = (|W_Z(\Lambda^+_{(m-4)n})| - 1)^{k^d}$. Hence, by Equation 3, $h(Y) > h(Z) - \epsilon$. Claim 1-■

Remark 5.7. A slight variation of this argument shows nontrivial FOSM SFTs have positive entropy. Though in Section 9 it is evident that the positive entropy can be deduced without reference to finite orbits. That is, non-trivial SM SFTs have positive entropy.

We will now show that Y is a FOSM SFT. Y definitely contains finite orbits, so we need only show Y is square mixing. To do this we pick a value k' for the square mixing gap parameter for Y and show the square mixing property holds.

Since Y is Λ_{M_0} scaled, let k' be large enough that for any $n' > 0$ there exists n with $n' < n < n' + k'$ such that for any $w \in \mathbb{Z}^d$, $w + \Lambda_{M_0}$ is a subset of either Λ_n, $\mathbb{Z}^d \backslash \Lambda_{n+k}$ or $\Lambda_{n'+k'} \backslash \Lambda_{n'}$ (*e.g.* let $k' = 4M_0 + 6 + k$ and $n = n' + 2M_0 + 2$). For $y, y' \in Y$, square mixing tells us there exists $z \in Z$ such that $z|_{\mathbb{Z}^d \backslash \Lambda_{n+k}} = y|_{\mathbb{Z}^d \backslash \Lambda_{n+k}}$ and $z|_{\Lambda_n} = y'|_{\Lambda_n}$.

Since $y \in Y$ we know no copies of W_0 occur in z on $\mathbb{Z}^d \backslash \Lambda_{n+k}$ (*i.e.* for $\Lambda_{M_0} + w \subset \mathbb{Z}^d \backslash \Lambda_{n+k}$, $y|_{\Lambda_{M_0}+w} \neq W_0$). Similarly no copies of W_0 occur in z on Λ_n. It is possible that copies of W_0 still occur in the annulus $\Lambda_{n'+k'} \backslash \Lambda_{n'}$. However, they may be removed. Suppose $\Lambda^+_{mn} + w \subset$

$\Lambda_{n'+k'}\backslash\Lambda_{n'}$ and $z|_{\Lambda^+_{mn}+w} = x|_{\Lambda^+_{mn}}$. Using the replacement procedure outlined before Lemma 5.5 we construct $\widehat{z} \in Z$. ($\widehat{z}|_{\mathbb{Z}^d\backslash\Lambda^+_{mn}+w} = z|_{\mathbb{Z}^d\backslash\Lambda^+_{mn}+w}$ and $\widehat{z}|_{\Lambda^+_{mn}+w} = V$.) By Lemma 5.5, $B(\widehat{z}) \subsetneq B(z)$, and since $B(z)$ is finite, $|B(\widehat{z})| < |B(z)|$. Moreover, $\widehat{z}|_{\mathbb{Z}^d\backslash\Lambda_{n'+k'}} = y|_{\mathbb{Z}^d\backslash\Lambda_{n'+k'}}$ and $\widehat{z}|_{\Lambda_{n'}} = y'|_{\Lambda_{n'}}$. Letting $\widehat{z}$ be renamed z and repeating this process a sufficient and finite number of times, we will have that $|B(z)| = 0$ (and hence $z \in Y$) and $z|_{\mathbb{Z}^d\backslash\Lambda_{n'+k'}} = y|_{\mathbb{Z}^d\backslash\Lambda_{n'+k'}}$ and $z|_{\Lambda_{n'}} = y'|_{\Lambda_{n'}}$. That is to say, Y is square mixing.

To complete the proof of Theorem 5.1, it remains to address the last sentence of 5.1. Since $Y' \subsetneq Z$ and the finite orbits are dense in Z there exists a point $x \in Z$ with a finite orbit such that $x \notin Y'$. So for N sufficiently large $x|_{\Lambda_N} \notin W_{Y'}(\Lambda_N)$. Let $W = x|_{\Lambda_N}$ and define $Y = \{z \in Z : y|_{\Lambda_N+v} \neq W, \forall v \in \mathbb{Z}^d\}$. Since $W \notin W_{Y'}(\Lambda_N)$, if $y \in Y'$, then $y|_{\Lambda_N+v} \neq W$ for all $v \in \mathbb{Z}^d$. That is $y \in Y$. So, we let x be the periodic point used at the beginning of the proof of 5.1 and we choose m and n large enough that $M_0 = mn \geq N$. ■

6. Constructing a Marker for the Image.

The primary purpose of this section is to prove Lemma 6.3. To make an injective map from X to Z we need a "marker" symbol in Z that we can use to "mark" the lattice elements $v \in \mathcal{M}_x$ in the target space Z. For Y and Z as in Theorem 5.1, the word W_0 itself may not be a marker for Y in Z, since it may fail to satisfy the *only if* portion of Part *i*) in the following definition.

Definition 6.1. Let Y and Z be subshifts such that $W_Y(\Lambda_M) \subsetneq W_Z(\Lambda_M)$ for all $M \geq M_0$ for some $M_0 \geq 0$. For $M \geq M_0$, a word $W \in W_Z(\Lambda_M)\backslash W_Y(\Lambda_M)$ is called a *marker for Y in Z* if there exists $M' \geq M$ such that for every $y \in Y$ there exists $z \in Z$ such that

i) $z|_{\Lambda_M+v} = W$ if and only if $v = 0$ and

ii) $z|_{\mathbb{Z}^d\backslash\Lambda_{M'}} = y|_{\mathbb{Z}^d\backslash\Lambda_{M'}}$.

If $M = 1$ then we refer to W as a marker *symbol* for Y in Z. We refer to M' as the *marker shadow size.*

Lemma 6.2. *Let Y and Z be subshifts such that $W_Y(\Lambda_M) \subsetneq W_Z(\Lambda_M)$ for all $M \geq M_0$ for some $M_0 \geq 0$. For $M \geq M_0$, if $W \in W_Z(\Lambda_M)\backslash W_Y(\Lambda_M)$ is a marker for Y in Z, then there are higher block recodings, $\widehat{Y}$ and $\widehat{Z}$, and a symbol $w \in W_{\widehat{Z}}(\Lambda_1)\backslash W_{\widehat{Y}}(\Lambda_1)$ which is a marker symbol for $\widehat{Y}$ in $\widehat{Z}$.*

Proof of 6.2. It suffices to recode using $W_Z(\Lambda_M)$. We let the symbol w correspond to the word W and one can check that *i)* and *ii)* (in Definition 6.1) hold with the recoded marker shadow size $\widehat{M'} = M + M' - 1$. ■

Lemma 6.3. *Let Z be a nontrivial square mixing SFT and for $M_0 > 0$ and $W_0 \in W_Z(\Lambda_{M_0})$ suppose $Y = \{z \in Z : z|_{\Lambda_{M_0}+v} \neq W_0 \ \forall v \in \mathbb{Z}^d\} \neq \emptyset$. Then, there exist $M_1 > M_0$ and a word $W_1 \in W_Z(\Lambda_{M_1})\backslash W_Y(\Lambda_{M_1})$ which is a marker for Y in Z.*

We will see that it suffices to let the marker shadow size $M' \geq k + M_1$, where k is the square mixing gap parameter for Z. Before we begin the proof of the lemma let us verify the following easy claim.

Claim. Suppose there exists $m > 0$ such that for every $x \in X$, $\sigma^p x = x$ for some $p \in \Lambda_m\backslash 0$. Then $h(X) = 0$.

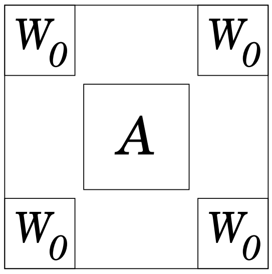
Proof. Let $X \subset S^{\mathbb{Z}^d}$ and let $X_p = \{x \in X : \sigma^p x = x\}$. Then $X = \bigcup_{p\in\Lambda_m\backslash 0} X_p$. It should be clear $h(X) \leq \max h(X_p)$ so we need only show $h(X_p) = 0$ for each $p \in \Lambda_m\backslash 0$. Fix $p \in \mathbb{Z}^d\backslash 0$. Regarding $p \in \mathbb{R}^d$, let H be the $d-1$ dimensional hyperplane perpendicular to p and suppose $\{q_i\}_{i=1}^{d-1}$ spans H. Let $\widehat{\Phi} = \{\sum r_i q_i : 0 \leq r_i \leq 1\}$. For $A \subset \mathbb{R}^d$ let $A \times p = \{rp + sq : 0 \leq r, s \leq 1 \text{ and } q \in A\}$ and for $n > 0$ let $nA = \{na : a \in A\}$. Then

$$h(X_p) \cdot Vol(\widehat{\Phi} \times p) = \lim_{n\to\infty} \frac{\log |W_{X_p}(n(\widehat{\Phi} \times p))|}{n^d} = \lim_{n\to\infty} \frac{\log |W_{X_p}(n\widehat{\Phi} \times p)|}{n^d}$$
$$\leq \lim_{n\to\infty} \frac{\log \left(|S|^{n^{d-1}Vol(\hat{\Phi}\times p)}\right)}{n^d} = 0.$$

■

Proof of 6.3. Assume Z has the square mixing parameter k. If there exists $m > 0$ such that for every $M > 0$ and $w \in W_Z(\Lambda_M)$, w is p periodic for some $p \in \Lambda_m\backslash 0$, then by compactness every $x \in Z$ has $\sigma^p x = x$ for some $p \in \Lambda_m\backslash 0$ and hence $h(Z) = 0$. Since Z is a nontrivial SM SFT, $h(Z) > 0$ (see Section 9) and so there exists an $m > 2(M_0+k+1)$ and a Z allowed word $A \in W_Z(\Lambda_m)$ which is not p periodic for any nonzero $p \in \Lambda_{2(M_0+k+1)}$. Since $A \in W_Z(\Lambda_m)$ there exists a point $\widehat{z} \in Z$ such that $\widehat{z}|_{\Lambda_m} = A$. Let $M_1 = m + 2M_0 + k + 1$. Because Z is a SM SFT there is a word $W_1 \in W_Z(\Lambda_{M_1})$ such that $W_1|_{\Lambda_m} = A$ and such that $W_1|_{\Lambda_{M_0}+v} = W_0$ for v equal to each $(p_1, .., p_d) \in \mathbb{Z}^d$ where each $p_i = \pm(M_1 - M_0)$. Figure 3 depicts W_1 for $d = 2$.

FIGURE 3. Constructing W_1.



Since $Y \subset Z$, for each point $y \in Y$ there exists a point $z \in Z$ such that $z = y$ outside Λ_{M_1+k} and $z|_{\Lambda_{M_1}} = W_1$. No other copy of W_1 exists in z since this would contradict $y \in Y$ (by the presence of W_0 in the region $\mathbb{Z}^d \backslash \Lambda_{M_1+k}$) or would contradict A not being p periodic for any $p \in \Lambda_{2(M_0+k+1)}$. ■

7. Constructing an Injection into Marked Words Using Entropy

In this section our principle result is Theorem 7.2, though we will actually use the closely related Corollary 7.2.1. For two subshifts X and Z, Corollary 7.2.1 gives sufficient conditions for the existence of an injection from the set of X allowed words into a set of Z allowed words on a suitable collection of polyhedrons. This will provide us with the means of guaranteeing the injective nature of the embedding which we seek in Theorem 2.5. Since the corollary follows almost directly from Theorem 7.2, most of this section will be spent proving the latter. In order to state Theorem 7.2 we need to define some terms.

For a convex polyhedron $\mathcal{V} \subset \mathbb{R}^d$ let $\mathcal{V}_K^+ \equiv \{r \in \mathbb{R}^d : \rho(r, \mathcal{V}) \leq K\}$ and let $\mathcal{V}_K^- \equiv \{r \in \mathcal{V} : \rho(r, \mathbb{R}^d \backslash \mathcal{V}) \geq K\}$. Let $\partial \mathcal{V}$ denote the boundary of $\mathcal{V}$ and let $\partial \mathcal{V}_K^+ \equiv \{r \in \mathbb{R}^d : \rho(r, \partial \mathcal{V}) \leq K\}$.

Definition 7.1. Let Z be a subshift and let $W \in W_Z(\Lambda_1)$. For a convex polyhedron $\mathcal{V}$ and a point $p \in \mathcal{V} \cap \mathbb{Z}^d$, define $W_Z(\mathcal{V}_1^+; p, W)$ to be the set of words $w \in W_Z(\mathcal{V}_1^+)$ such that (for $i \in \mathcal{V}_1^+$)

$$w_i = W \text{ if and only if } i = p.$$

For $u \in W_Z(\partial \mathcal{V}_1^+)$ define

$$W_Z^u(\mathcal{V}_1^+) \equiv \{w \in W_Z(\mathcal{V}_1^+) : w|_{\partial \mathcal{V}_1^+} = u\}.$$

We think of $W_Z(\mathcal{V}_1^+; p, W)$ as the collection of words in $W_Z(\mathcal{V}_1^+)$ which have been "marked" at p with the symbol W and we think of $W_Z^u(\mathcal{V}_1^+)$ as the collection of words in $W_Z(\mathcal{V}_1^+)$ which look like u on the neighborhood $\partial \mathcal{V}_1^+$ of the boundary of $\mathcal{V}$.

For $i_0 > 0$ we say a family $\bigcup_{i \geq i_0} \mathfrak{F}(i)$ is a 2 *uniform family* of convex polyhedrons if each $\mathfrak{F}(i)$ is a set of convex polyhedrons which, for $\mathcal{V} \in \mathfrak{F}(i)$, we have $B(p, i/2) \subset \mathcal{V} \subset B(q, 2i)$ for some $p, q \in \mathbb{R}^d$.

Theorem 7.2. *Let Y and Z be nontrivial SM subshifts such that $W_Y(\Lambda_m) \subsetneq W_Z(\Lambda_m)$ for all $m \geq 0$ and let $W \in W_Z(\Lambda_1) \backslash W_Y(\Lambda_1)$ be a marker symbol for Y in Z. For any ϵ such that $0 < \epsilon < h(Y)$, there exists $m_2 \geq 0$ such that for every 2 uniform family of convex polyhedrons $\bigcup_{i \geq i_0} \mathfrak{F}(i)$, $i \geq m_2$, $\mathcal{V} \in \mathfrak{F}(i)$ with $B(0, i/2) \subset \mathcal{V}$ and, $u \in W_Y(\partial \mathcal{V}_1^+)$ we have*

$$e^{(h(Y)-\epsilon) Vol(\mathcal{V})} < |W_Z^u(\mathcal{V}_1^+) \cap W_Z(\mathcal{V}_1^+; 0, W)|.$$

Corollary 7.2.1. *Let Y and Z be nontrivial SM subshifts such that $W_Y(\Lambda_m) \subsetneq W_Z(\Lambda_m)$ for all $m \geq 0$ and let $W \in W_Z(\Lambda_1) \backslash W_Y(\Lambda_1)$ be a marker symbol for Y in Z. For any subshift X with $h(X) < h(Y)$, there exists $m_2 \geq 0$ such that for every 2 uniform family of convex polyhedrons $\bigcup_{i \geq i_0} \mathfrak{F}(i)$, $i \geq m_2$, $\mathcal{V} \in \mathfrak{F}(i)$ with $B(0, i/2) \subset \mathcal{V}$ and, $u \in W_Y(\partial \mathcal{V}_1^+)$ there is an injection*

$$\Psi(u) : W_X(\mathcal{V}) \hookrightarrow W_Z^u(\mathcal{V}_1^+) \cap W_Z(\mathcal{V}_1^+; 0, W). \tag{6}$$

Corollary 7.2.1 follows directly from Theorem 7.2 and Lemma 7.4 below.

Since we identify $W_{\mathcal{A}}(B)$ with $W_{\mathcal{A}}(B-v) = \sigma^v(W_{\mathcal{A}}(B))$ we will extend the definition of Ψ allowing it to take values from $\sigma^v W_Y(\partial \mathcal{V}_1^+) = W_Y(\sigma^v \partial \mathcal{V}_1^+)$ and $\sigma^v W_X(\mathcal{V}) = W_X(\sigma^v \mathcal{V})$ to $\sigma^v\big(W_Z^u(\mathcal{V}_1^+) \cap W_Z(\mathcal{V}_1^+; 0, W)\big) = W_Z^{\sigma^v u}(\sigma^v \mathcal{V}_1^+) \cap W_Z(\sigma^v \mathcal{V}_1^+; -v, W)$. That is, for $\sigma^v u \in W_Y(\sigma^v \partial \mathcal{V}_1^+)$

$$\Psi(\sigma^v u) : W_X(\sigma^v \mathcal{V}) \hookrightarrow W_Z^{\sigma^v u}(\sigma^v \mathcal{V}_1^+) \cap W_Z(\sigma^v \mathcal{V}_1^+; -v, W).$$

where

$$\Psi(\sigma^v u)(\sigma^v w) = \sigma^v \Psi(u)(w).$$

Remark 7.3. Given $\epsilon > 0$ and $K > 0$, there exists $i_1 > 0$ such that for any 2 uniform family of convex polyhedrons $\bigcup_{i \geq i_0} \mathfrak{F}(i)$, $i \geq i_1$, and $\mathcal{V} \in \mathfrak{F}(i)$ we have

$$(1-\epsilon) Vol(\mathcal{V}) \leq Vol(\mathcal{V}_K^-) \text{ and } Vol(\mathcal{V}_K^+) \leq (1+\epsilon) Vol(\mathcal{V}).$$

Lemma 7.4. *Let X be a $\mathbb{Z}^d$ subshift and let $\epsilon > 0$. There exists $i_1 > 0$ such that for any 2 uniform family of convex polyhedrons $\bigcup_{i \geq i_o} \mathfrak{F}(i)$, $i \geq i_1$, and $\mathcal{V} \in \mathfrak{F}(i)$ we have $|W_X(\mathcal{V})| \leq e^{(h(X)+\epsilon) Vol(\mathcal{V})}$.*

Proof of 7.4. From the definition of entropy, there exists N such that $|W_X(\Lambda_N^+)| \leq e^{(h(X)+\epsilon) Vol(\Lambda_N^+)}$. By Remark 7.3, for $K = \sqrt{d} \cdot N$ there exists i_1 such that for $i \geq i_1$ and $\mathcal{V} \in \mathfrak{F}(i)$ we have $Vol(\mathcal{V}_K^+) \leq (1+\epsilon) Vol(\mathcal{V})$. We cover $\mathcal{V}$ with disjoint d-cubes $\Lambda_N^+ + v$. Suppose the minimal number of d-cubes necessary for this is L. Then we know $\dfrac{L \cdot Vol(\Lambda_N^+)}{Vol(\mathcal{V})} \leq \dfrac{Vol(\mathcal{V}_K^+)}{Vol(\mathcal{V})} \leq 1+\epsilon$. Putting this together we get

$$\begin{aligned}
\frac{\log |W_X(\mathcal{V})|}{Vol(\mathcal{V})} &\leq \frac{L \cdot \log \big|W_X(\Lambda_N^+)\big|}{Vol(\mathcal{V})} \cdot \frac{Vol(\Lambda_N^+)}{Vol(\Lambda_N^+)} \\
&\leq \frac{\log \big|W_X(\Lambda_N^+)\big|}{Vol(\Lambda_N^+)} (1+\epsilon) \\
&\leq (h(X)+\epsilon)(1+\epsilon) \leq h(X) + \epsilon'.
\end{aligned}$$

■

The following lemma is a simpler version of Theorem 7.2. We present it so that we may preview our approach to the proof before proving 7.2.

Lemma 7.5. *Let Y be a nontrivial $\mathbb{Z}^d$ SM subshift and $\mathfrak{F} = \bigcup_{i \geq i_o} \mathfrak{F}(i)$ a 2 uniform family of convex polyhedrons. Given $\epsilon > 0$ there exists i_1 such that for $i \geq i_1$ and $\mathcal{V} \in \mathfrak{F}(i)$, we have $|W_Y(\mathcal{V})| \geq e^{(h(Y)-\epsilon)Vol(\mathcal{V})}$.*

Proof of 7.5. Let $k \geq k_Y$ where k_Y is the square mixing parameter for Y. By the definition of entropy, for $N \geq k$ suitably large, $|W_Y(\Lambda_{N-k})| \geq e^{(h(Y)-\epsilon)Vol(\Lambda_N)}$. For $N' = 2N$ and $e = (1, .., 1) \in \mathbb{Z}^d$, $\Lambda_N \subset \Lambda^+_{N'} - Ne$ and $Vol(\Lambda_N) = Vol(\Lambda^+_{N'})$. By Remark 7.3, for $K = \sqrt{d} \cdot N'$, there exists i_1 such that for $i \geq i_1$ and $\mathcal{V} \in \mathfrak{F}(i)$, we have $(1-\epsilon)Vol(\mathcal{V}) \leq Vol(\mathcal{V}^-_K)$.

Since the collection $\{\Lambda^+_{N'} - Ne + N'v\}_{v \in \mathbb{Z}^d}$ is a cover of $\mathbb{R}^d$ by disjoint sets, for a convex polyhedron $\mathcal{V}$ and $K = \sqrt{d} \cdot N'$ there exists a finite set $A \subset \mathbb{Z}^d$ such that the set $\{\Lambda^+_{N'} - Ne + N'v\}_{v \in A}$ is a cover of $\mathcal{V}^-_K$ (by disjoint sets) and each $\Lambda^+_{N'} - Ne + N'v \subset \mathcal{V}$ for $v \in A$. So $\{\Lambda_N + N'v\}_{v \in A}$ is a collection of disjoint sets and each $\Lambda_N + N'v \subset \mathcal{V}$.

Square mixing guarantees that for each $\eta \in W_Y(\Lambda_{N-k})^A$, there exists a word $w \in W_Y(\mathcal{V})$ such that $w|_{\Lambda_{N-k}+N'v} = \eta_v$. Thus $|W_Y(\mathcal{V})| \geq |W_Y(\Lambda_{N-k})|^L$ where $L = |A|$. Since $\cup_{v \in A}(\Lambda^+_{N'} - Ne + N'v) \subset \mathcal{V}$ is a cover of $\mathcal{V}^-_K$ by disjoint half-open d-cubes we have $\frac{L \cdot Vol(\Lambda^+_{N'})}{Vol(\mathcal{V})} \geq \frac{Vol(\mathcal{V}^-_K)}{Vol(\mathcal{V})} \geq 1 - \epsilon$. Putting these facts together we get

$$\begin{aligned} \frac{\log |W_Y(\mathcal{V})|}{Vol(\mathcal{V})} &\geq \frac{L \cdot \log |W_Y(\Lambda_{N-k})|}{Vol(\mathcal{V})} \cdot \frac{Vol(\Lambda_N)}{Vol(\Lambda_N)} \cdot \frac{Vol(\Lambda^+_{N'})}{Vol(\Lambda^+_{N'})} \\ &\geq \frac{\log |W_Y(\Lambda_{N-k})|}{Vol(\Lambda_N)}(1-\epsilon) \\ &\geq (h(Y)-\epsilon)(1-\epsilon) \geq h(Y) - \epsilon'. \end{aligned}$$

■

Proof of 7.2. Since W is a marker for Y in Z it has a marker shadow size M'. Let $k \geq k_Y$ and suppose $N \geq M'$ is large enough that $|W_Y(\Lambda_{N-k})| \geq e^{(h(Y)-\epsilon)Vol(\Lambda_N)}$. Let $N' = 2N$. By Remark 7.3, for $K = \sqrt{d} \cdot N' + 2$, there exists i_1 such that for $i \geq i_1$ and

$\mathcal{V} \in \mathfrak{F}(i)$, we have $(1-\epsilon)Vol(\mathcal{V}) \leq Vol(\mathcal{V}_K^-)$. For $i_1 \geq 2(\sqrt{d} \cdot N + 3)$, $i \geq i_1$ and $\mathcal{V} \in \mathfrak{F}(i)$, if $B(0, i/2) \subset \mathcal{V}$, then $\Lambda_N \subset \mathcal{V}_2^-$.

There exists a set $A \subset \mathbb{Z}^d$ such that 1) the collection $\{\Lambda_{N'}^+ - Ne + N'v\}_{v\in A}$ is a cover of $\mathcal{V}_K^-$ (by disjoint sets) and each $\Lambda_{N'}^+ - Ne + N'v \subset \mathcal{V}_2^-$ for $v \in A$, and 2) the collection $\{\Lambda_N + N'v\}_{v \in A}$ is a collection of disjoint sets such that for each $v \in A$, $\Lambda_N + N'v \subset \mathcal{V}_2^-$. We may arrange to have $0 \in A$. (It is not inconceivable that $0 \notin A$, but we can choose A such that $0 \in A$ without any impact on the foregoing.) Moreover (letting $L = |A|$), for i_1 large enough, $i \geq i_1$ and $\mathcal{V} \in \mathcal{F}(i)$ we have $\frac{L-1}{L} > 1 - \epsilon$.

By 1) we have $\frac{L \cdot |Vol(\Lambda_{N'}^+)|}{Vol(\mathcal{V})} \geq \frac{Vol(\mathcal{V}_K^-)}{Vol(\mathcal{V})} \geq 1-\epsilon$. For $i \geq i_1$ and $\mathcal{V} \in \mathfrak{F}(i)$ with $B(0, i/2) \subset \mathcal{V}$, by square mixing and 2), for every $u \in W_Y(\partial \mathcal{V}_1^+)$ and $\eta \in W_Y(\Lambda_{N-k})^{A \backslash 0}$ there exists a word $w \in W_Y(\mathcal{V}_1^+)$ such that $w|_{\partial \mathcal{V}_1^+} = u$, and $w|_{\Lambda_{N-k}+N'v} = \eta_v$ for all $v \in A \backslash 0$. That is, $w \in W_Y^u(\mathcal{V}_1^+)$ and there is an injection from $W_Y(\Lambda_{N-k})^{A\backslash 0}$ into $W_Y^u(\mathcal{V}_1^+)$. Since W is a marker symbol for Y in Z, $N \geq M'$, and $\Lambda_{M'} \subset \mathcal{V}_2^- \backslash \cup_{v \in A \backslash 0} (\Lambda_N + N'v)$ there exists $w' \in W_Z(\mathcal{V}_1^+)$ such that $w' = w$ on $\mathcal{V}_1^+ \backslash \Lambda_{M'}$ and $w'_i = W \iff i = 0$. That is, $w' \in W_Z(\mathcal{V}_1^+; 0, W)$. Because $w' = w$ on $\mathcal{V}_1^+ \backslash \Lambda_{M'}$, $w' \in W_Z(\mathcal{V}_1^+; 0, W) \cap W_Z^u(\mathcal{V}_1^+)$ and we have an injection from $W_Y(\Lambda_{N-k})^{A \backslash 0}$ into $W_Z(\mathcal{V}_1^+; 0, W) \cap W_Z^u(\mathcal{V}_1^+)$. So $|W_Z(\mathcal{V}_1^+; 0, W) \cap W_Z^u(\mathcal{V}_1^+)| \geq |W_Y(\Lambda_{N-k})|^{L-1}$. Thus we have

$$\begin{aligned}
\frac{\log \left| W_Z^u(\mathcal{V}_1^+) \cap W_Z(\mathcal{V}_1^+; 0, W)) \right|}{Vol(\mathcal{V})} &\geq \frac{(L-1) \cdot \log |W_Y(\Lambda_{N-k})|}{Vol(\mathcal{V})} \cdot \frac{L \cdot Vol(\Lambda_N)}{L \cdot Vol(\Lambda_N)} \cdot \frac{Vol(\Lambda_{N'}^+)}{Vol(\Lambda_{N'}^+)} \\
&\geq \frac{\log |W_Y(\Lambda_{N-k})|}{Vol(\Lambda_N)} (1-\epsilon)^2 \\
&\geq h(Y) - \epsilon''.
\end{aligned}$$

■

8. Constructing a $\mathbb{Z}^d$ Embedding: The Proof.

We are now ready to complete the proof of Theorem 2.5, demonstrating the sufficiency of the entropy gap (outlined at the end of Section 2). We assume the conditions of 2.5. Namely, let X be a non-periodic $\mathbb{Z}^d$ subshift, Z a $\mathbb{Z}^d$ FOSM SFT, and $\phi : X \to Z$ a homomorphism.

Suppose $h(X) < h(Z)$ and $\epsilon > 0$ is such that $h(X) + \epsilon < h(Z) - \epsilon$; we will produce an embedding $\psi : X \hookrightarrow Z$.

Since Z is a nontrivial FOSM SFT, by Theorem 5.1 there exist M_0 and $W_0 \in W_Z(\Lambda_{M_0})$ such that $Y = \{z \in Z : z|_{\Lambda_{M_0}+v} \neq W_0 \; \forall \, v \in \mathbb{Z}^d\}$ is a nonempty FOSM SFT with $Y \subsetneq Z$ and $h(Y) > h(Z) - \epsilon$. Moreover, since $\phi(X) \subsetneq Z$ is a subshift we may assume $\phi(X) \subset Y$.

By Lemma 6.3 there exist M_1 and a word $W_1 \in W_Z(\Lambda_{M_1}) \backslash W_Y(\Lambda_{M_1})$ such that W_1 is a marker for Y in Z. By Lemma 6.2 we may recode using $W_Z(\Lambda_{M_1})$ to form the conjugate subshifts $\widehat{Y}$ and $\widehat{Z}$ and the marker symbol $\widehat{W_1} \in W_{\widehat{Z}}(\Lambda_1) \backslash W_{\widehat{Y}}(\Lambda_1)$ for $\widehat{Y}$ in $\widehat{Z}$. Moreover, picking M_1 large enough we may assume $\widehat{Y}$ and $\widehat{Z}$ are matrix SFTs. Since $\widehat{Y}$ is conjugate to Y there is a homomorphism $\widehat{\phi} : X \to \widehat{Y}$ and $h(\widehat{Y}) = h(Y)$, and likewise $h(Z) = h(\widehat{Z})$. Dropping the "hat" we have that Y and Z are matrix FOSM SFTs with $Y \subsetneq Z$, $h(X) < h(Y)$, $\phi : X \to Y$ is a homomorphism and $W_1 \in W_Z(\Lambda_1) \backslash W_Y(\Lambda_1)$ is a marker symbol for Y in Z with marker shadow M'.

By Corollary 7.2.1, there exists m_2 such that for every 2 uniform family of convex polyhedrons $\cup \mathfrak{F}(i)$, $m \geq m_2$, $\mathcal{V} \in \mathfrak{F}(m)$ with $B(0, m/2) \subset \mathcal{V}$ and $u \in W_Y(\partial \mathcal{V}_1^+)$ there is an injection $\Psi(u) : W_X(\mathcal{V}) \hookrightarrow W_Z^u(\mathcal{V}_1^+) \cap W_Z(\mathcal{V}_1^+; 0, W_1)$. For $m \geq \max\{m_2, \sqrt{d}/2\}$ fixed, according to Lemma 4.4, there exist M and a clopen set $F(m, M) \subset X$ (which we now fix) such that for each $x \in X$ the set $\mathcal{M}_x \in \mathfrak{M}_m$. By Remark 4.3 the map $x \mapsto \mathcal{M}_x$ is continuous and shift commuting.

By Lemma 3.5 we may conclude that $x \mapsto \mathfrak{B}_m(\mathcal{M}_x)$ is a continuous shift commuting map, $\mathfrak{B}_m(\mathcal{M}_x)$ is a regular $\mathfrak{F}_m$ covering of $\mathbb{R}^d$ and for each $(\mathcal{V}, v) \in \mathfrak{B}_m(\mathcal{M}_x)$, $v \in \mathcal{M}_x$ and $\mathcal{V}$ is a convex polyhedron with $\overline{B}(0, m/2) \subset \mathcal{V} \subset B(0, 2m)$. For each $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)$, the word $\phi(x)|_{\partial \mathcal{V}_1^+ + v} \in W_Y(\partial \mathcal{V}_1^+ + v)$ and $x|_{\mathcal{V}+v} \in W_X(\mathcal{V} + v)$. Thus, the word $\Psi(\phi(x)|_{\partial \mathcal{V}_1^+ + v})(x|_{\mathcal{V}+v}) \in W_Z(\mathcal{V}_1^+ + v; v, W_1) \cap W_Z^u(\mathcal{V}_1^+ + v)$ where $u = \phi(x)|_{\partial \mathcal{V}_1^+ + v}$. Because of this we may make the following definition.

Definition 8.1. *For each $x \in X$ and $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)$ define*

$$\psi(x)|_{\mathcal{V}_1^+ + v} \equiv \Psi(\phi(x)|_{\partial \mathcal{V}_1^+ + v})(x|_{\mathcal{V}+v}).$$

ψ will prove to be the embedding desired in Theorem 2.5 which we shall show by means of three lemmas.

Lemma 8.2. *For each $x \in X$, $\psi(x)$ is a well-defined point in Z.*

Proof of 8.2. To show $\psi(x)$ is well defined suppose $(\mathcal{V}, v), (\mathcal{V}', v') \in \mathfrak{V}_m(\mathcal{M}_x)$ are distinct; thus $\mathcal{V}_1^+ + v \cap \mathcal{V}_1'^+ + v' = \partial\mathcal{V}_1^+ + v \cap \partial\mathcal{V}_1'^+ + v'$. Because $\psi(x)|_{\partial\mathcal{V}_1^+ + v} = \phi(x)|_{\partial\mathcal{V}_1^+ + v}$ for $(\mathcal{V}, v) \in \mathfrak{B}_m(\mathcal{M}_x)$ we have

$$\psi(x)|_{\mathcal{V}_1^+ + v}|_{\mathcal{V}_1'^+ + v'} = \phi(x)|_{\partial\mathcal{V}_1^+ + v \cap \partial\mathcal{V}_1'^+ + v'} = \psi(x)|_{\mathcal{V}_1'^+ + v'}|_{\mathcal{V}_1^+ + v}.$$

Thus $\psi(x)$ is well defined.

Because Z is presented by matrices and because each adjacent pair of vertices (*e.g.* v and $v + e_i$ for $1 \le i \le d$) is contained in $\mathcal{V}_1^+ + v$ for some $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)$, it follows that $\psi(x) \in Z$ since each $\psi(x)|_{\mathcal{V}_1^+ + v}$ is Z allowed. ■

Lemma 8.3. *The map $\psi : X \to Z$ is shift commuting and continuous.*

Proof of 8.3. We begin with the commutativity; $\psi(\sigma^v x) = \sigma^v \psi(x)$. Observe $(\mathcal{V}, u) \in \mathfrak{V}_m(\mathcal{M}_x) \iff (\mathcal{V}, u - v) \in \mathfrak{V}_m(\mathcal{M}_{\sigma^v x})$. Recall, $\sigma^v z|_{\sigma^v \mathcal{A}} = \sigma^v(z|_{\mathcal{A}})$ and $\sigma^v \mathcal{A} = \mathcal{A} - v$. So for $(\mathcal{V}, u) \in \mathfrak{V}_m(\mathcal{M}_x)$

$$\begin{aligned}(\sigma^{-v}\psi(\sigma^v x))|_{\mathcal{V}_1^+ + u} = \sigma^{-v}\left(\psi(\sigma^v x)|_{\mathcal{V}_1^+ + u - v}\right) &\equiv \sigma^{-v}\left(\Psi(\phi(\sigma^v x)|_{\partial\mathcal{V}_1^+ + u - v})\left(\sigma^v x|_{\mathcal{V} + u - v}\right)\right)\\ &= \sigma^{-v}\left(\Psi(\sigma^v(\phi(x)|_{\partial\mathcal{V}_1^+ + u}))\left(\sigma^v(x|_{\mathcal{V}+u})\right)\right)\\ &= \Psi(\phi(x)|_{\partial\mathcal{V}_1^+ + v})\left(x|_{\mathcal{V}+u}\right)\\ &\equiv \psi(x)|_{\mathcal{V}_1^+ + u}\end{aligned}$$

or $\psi(\sigma^v x) = \sigma^v \psi(x)$.

Continuity. Given k there exists $k_2 > k$ such that if $x|_{\Lambda_{k_2}} = y|_{\Lambda_{k_2}}$ then $\phi(x)$ and $\phi(y)$ agree on Λ_{k+4m+1}. By the continuity of the map $x \mapsto \mathfrak{V}_m(\mathcal{M}_x)$ we know there exists $k_3 \ge k$ such that if $x|_{\Lambda_{k_3}} = y|_{\Lambda_{k_3}}$ then $\mathfrak{V}_m(\mathcal{M}_x)$ and $\mathfrak{V}_m(\mathcal{M}_y)$ agree on $\mathfrak{F}_m \times \Lambda_{k+2m}$. Hence for $k_4 \ge \max\{k_2, k_3, k+4m+1\}$, x and y with $x|_{\Lambda_{k_4}} = y|_{\Lambda_{k_4}}$ and $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)|_{\mathfrak{F}_m \times \Lambda_{k+2m}} =$

$\mathfrak{V}_m(\mathcal{M}_y)|_{\mathfrak{F}_m\times\Lambda_{k+2m}}$ we have that $\phi(x)|_{\mathcal{V}_1^+ + v} = \phi(y)|_{\mathcal{V}_1^+ + v}$. Since $x|_{\Lambda_{k+4m+1}} = y|_{\Lambda_{k+4m+1}}$, then $x|_{\mathcal{V}+v} = y|_{\mathcal{V}+v}$ and hence

$$\psi(x)|_{\mathcal{V}_1^+ + v} = \Psi(\phi(x)|_{\partial\mathcal{V}_1^+ + v})(x|_{\mathcal{V}+v}) = \Psi(\phi(y)|_{\partial\mathcal{V}_1^+ + v})(y|_{\mathcal{V}+v}) = \psi(y)|_{\mathcal{V}_1^+ + v}.$$

Thus $\psi(x)$ and $\psi(y)$ agree on Λ_k. ■

Let us address the injectivity with the following lemma.

Lemma 8.4. *Given $u \in \mathbb{Z}^d$ and $x, y \in X$, if $x_u \neq y_u$ then there exists $w \in B(u, 6m)$ such that $\psi(x)_w \neq \psi(y)_w$. Hence ψ is an injection.*

Proof of 8.4. First, suppose $\mathcal{M}_x \cap \overline{B}(u, 6m) \neq \mathcal{M}_y \cap \overline{B}(u, 6m)$. Specifically, suppose $v \in \mathcal{M}_x \cap \overline{B}(u, 6m)$, but $v \notin \mathcal{M}_y \cap \overline{B}(u, 6m)$. From the definition of $\psi(x)$ we have $w \in \mathcal{M}_x \iff \psi(x)_w = W_1$ thus $\psi(x)_v = W_1 \neq \psi(y)_v$.

On the other hand, suppose $\mathcal{M}_x \cap \overline{B}(u, 6m) = \mathcal{M}_y \cap \overline{B}(u, 6m)$. Then $u \in (\mathcal{V} + v)$ for some $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)$. We *claim*: $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_y)$, as well. To see this, observe *i)* $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_x)$ implies $v \in \mathcal{M}_x$ and *ii)* the Voronoi tile $\mathcal{V}_v$ associated to v with respect to $\mathcal{M}_x$ is determined by $\mathcal{M}_x \cap \overline{B}(v, 4m)$ and $\mathcal{V}_v = \mathcal{V} + v$. Now, because $d(u, v) \leq 2m$ it follows that $\mathcal{M}_x \cap \overline{B}(v, 4m) = \mathcal{M}_y \cap \overline{B}(v, 4m)$ which implies *i)* $v \in \mathcal{M}_y$ and *ii)* since the Voronoi tile $\mathcal{V}'_v$ associated to v with respect to $\mathcal{M}_y$ is determined by $\mathcal{M}_y \cap \overline{B}(v, 4m) = \mathcal{M}_x \cap \overline{B}(v, 4m)$, that $\mathcal{V}'_v = \mathcal{V} + v$. Thus $(\mathcal{V}, v) \in \mathfrak{V}_m(\mathcal{M}_y)$.

Either $\phi(x)|_{\partial\mathcal{V}_1^+ + v} = \phi(y)|_{\partial\mathcal{V}_1^+ + v}$ or not. If not, then

$$\psi(x)|_{\partial\mathcal{V}_1^+ + v} = \Psi(\phi(x)|_{\partial\mathcal{V}_1^+ + v})(x|_{\mathcal{V}+v})|_{\partial\mathcal{V}_1^+ + v} = \phi(x)|_{\partial\mathcal{V}_1^+ + v} \neq \phi(y)|_{\partial\mathcal{V}_1^+ + v} = \psi(y)|_{\partial\mathcal{V}_1^+ + v}.$$

If so, then since $x|_{\mathcal{V}+v} \neq y|_{\mathcal{V}+v}$ we may now check

$$\begin{aligned} \psi(x)|_{\mathcal{V}_1^+ + v} &= \Psi(\phi(x)|_{\partial\mathcal{V}_1^+ + v})(x|_{\mathcal{V}+v}) \\ &\neq \Psi(\phi(x)|_{\partial\mathcal{V}_1^+ + v})(y|_{\mathcal{V}+v}) \\ &= \Psi(\phi(y)|_{\partial\mathcal{V}_1^+ + v})(y|_{\mathcal{V}+v}) \\ &= \psi(y)|_{\mathcal{V}_1^+ + v}. \end{aligned}$$

Thus for some $w \in \mathcal{V}_1^+ + v \subset \overline{B}(u, 6m)$ we have $\psi(x)_w \neq \psi(y)_w$ as we wanted to show. ■

In conclusion ψ is a continuous shift commuting injection from X into Z which completes the proof of the sufficiency of the conditions in Theorem 2.5. We recall, when addressing the necessity of the entropy gap, that periodic point considerations guarantee that any embedded image $\phi(X) \subset Z$ of a non-periodic subshift X into a square mixing SFT Z will always be a proper embedding. The entropy minimality of square mixing SFTs will then allow us to conclude $h(X) < h(Z)$, thus finishing the proof of Theorem 2.5.

9. SM SFTs and Their Properties.

Lemma 9.1. *A SM SFT is entropy minimal.*

Proof: Let X be a SM matrix SFT. Let $B_n \equiv \Lambda_{n+1}^+ \cap \mathbb{Z}^d$ be the d-cube in $\mathbb{Z}^d$ with sides of length n. We can rephrase the definition of square mixing with a new value k such that for $n \geq k$ and $x, y \in X$ there exists $z \in X$ such that $y|_{\mathbb{Z}^d \setminus B_n} = z|_{\mathbb{Z}^d \setminus B_n}$ and $x|_{B_{n-2k}+ke} = z|_{B_{n-2k}+ke}$ where $e = (1, .., 1)$.

If we have a subshift $X' \subsetneq X$ there exists a word $\alpha \in W_X(B_m) \setminus W_{X'}(B_m)$ for some $m > 0$. Let $K \equiv 2(k+1)+m$ be an integer and let $\epsilon < \frac{1}{|W_X(B_K)|}$. Let $\partial B_K \equiv B_K \setminus (B_{K-2}+e)$ and let $v^+ \equiv v+e$. For any $\mathcal{A} \subset \mathbb{Z}^d$ such that $B_K + v \subset \mathcal{A}$ for some $v \in \mathbb{Z}^d$ let $\mathcal{C} \equiv \mathcal{A} \setminus (B_{K-2} + v^+)$. That is $\mathcal{C} \cup (B_K + v) = \mathcal{A}$ and $\mathcal{C} \cap (B_K + v) = \partial B_K + v$.

For each $\partial u \in W_X(\partial B_K)$ let $W_X^{\partial u}(\mathcal{C}) = \{w \in W_X(\mathcal{C}) : w|_{\partial B_K + v} = \partial u\}$. Similarly, let $W_X^{\partial u}(B_K) \equiv \{w \in W_X(B_K) : w|_{\partial B_K} = \partial u\}$. Then

$$|W_X(\mathcal{A})| = \sum_{\partial u \in W_X(\partial B_K)} |W_X^{\partial u}(B_K)| \cdot |W_X^{\partial u}(\mathcal{C})|$$

and similarly

$$|W_{X'}(\mathcal{A})| \leq \sum_{\partial u \in W_{X'}(\partial B_K)} |W_{X'}^{\partial u}(B_K)| \cdot |W_{X'}^{\partial u}(\mathcal{C})|.$$

For any $\partial u \in W_{X'}(\partial B_K)$ there exists $u \in W_X^{\partial u}(B_K)$ such that $u|_{B_m+(k+1)e} = \alpha$ due to square mixing. Thus we have

$$|W_{X'}^{\partial u}(B_K)| \leq |W_X^{\partial u}(B_K)| - 1 < |W_X^{\partial u}(B_K)|(1 - \epsilon).$$

Since $W_{X'}(\partial B_K) \subset W_X(\partial B_K)$ the above inequalities yield $|W_{X'}(\mathcal{A})| \leq |W_X(\mathcal{A})|(1-\epsilon)$.

Let $M = Kn$, let F and G be a partition of $\Lambda_n^+ \cap \mathbb{Z}^d$ (*i.e.* $F \cup G = \Lambda_n^+ \cap \mathbb{Z}^d$ and $F \cap G = \emptyset$). Observe $\{B_K + Kv\}_{v \in \Lambda_n^+ \cap \mathbb{Z}^d}$ is a cover of B_M where the intersection of distinct elements contains at most elements of the boundaries of the sets $B_K + Kv$. Square mixing implies that for any word $w' \in W_X(\bigcup_{v \in G} B_K + Kv)$ and any function $f : F \to W_X(B_m)$, there exists a word $w \in W_X(B_M)$ such that $w|_{B_m+(k+1)e+Kv} = f(v)$ for all $v \in F$ and $w|_{B_K+Kv} = w'|_{B_K+Kv}$ for all $v \in G$.

Index the elements of $\Lambda_n^+ \cap \mathbb{Z}^d$ as v_i for $i = 1$ to n^d and define $G_0 \equiv \emptyset$ and $G_j \equiv \bigcup_{i=1}^{j} \{v_i\}$ for $j = 1$ to n^d. So $G_{j-1} \subsetneq G_j$ and $G_{n^d} = \Lambda_n^+ \cap \mathbb{Z}^d$. Let $F_j \equiv (\Lambda_n^+ \cap \mathbb{Z}^d) \backslash G_j$. Define

$$W_{X'}[G_j] \equiv \{w \in W_X(B_M) : w \text{ is } X' \text{ allowed on } \bigcup_{v \in G_j} B_K + Kv\}.$$

Now $G_j = G_{j-1} \cup \{v_j\}$, so

$$W_{X'}[G_j] \equiv \{w \in W_{X'}[G_{j-1}] : w \text{ is } X' \text{ allowed on } B_K + Kv_j\}.$$

And so like the calculation for $\mathcal{A}$ above we have

$$|W_{X'}[G_j]| \leq (1-\epsilon)|W_{X'}[G_{j-1}]|.$$

Thus

$$|W_{X'}(B_M)| \leq (1-\epsilon)^{n^d} |W_X(B_M)|.$$

Hence

$$\frac{\log |W_{X'}(B_M)|}{M^d} \leq \frac{n^d \log(1-\epsilon)}{n^d K^d} + \frac{\log |W_X(B_M)|}{M^d}$$

and thus $h(X') \leq \frac{\log(1-\epsilon)}{K^d} + h(X) < h(X)$. ∎

The proof of the following lemma does not extend to $d > 2$. Whether the lemma holds for $d > 2$ is an open question.

Lemma 9.2. *A $\mathbb{Z}^2$ square mixing SFT contains a finite orbit.*

Proof. Let $H^- = \{(k,l) \in \mathbb{Z}^2 : k < 0\}$, $H^+ = \mathbb{Z}^2 \backslash H^-$ and $V = \{(0,k) \in \mathbb{Z}^2 : k \in \mathbb{Z}\}$. Let X be a SM matrix SFT with square mixing gap parameter k.

Claim. For any $x, y \in X$ there exists $z \in X$ such that $z|_{H^-} = x|_{H^-}$ and $z|_{H^+ + (k,0)} = y|_{H^+ + (k,0)}$.

To prove this we observe that by square mixing for $n \geq k$ there exists a point $z_n \in X$ such that $z_n|_{\mathbb{Z}^2 \backslash (\Lambda_{n+1} + (n,0))} = x|_{\mathbb{Z}^2 \backslash (\Lambda_{n+1} + (n,0))}$ and $z_n|_{\Lambda_{n+1-k} + (n,0)} = y|_{\Lambda_{n+1-k} + (n,0)}$. By compactness, the set $\{z_n\}_{n \geq N}$ has an accumulation point z and one can check that such a point z satisfies the claim.

Now given y and $x = \sigma^{(k+1,0)} y$ the point z (given by the claim) has the property that $z|_{V-(1,0)} = z|_{V+(k,0)}$. Let $L(j) = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i \leq k\}$. The set

$$H^+ \backslash (H^+ + (k+1,0)) = \bigcup_{j \in \mathbb{Z}} L(j)$$

is an infinite vertical strip. So $z|_{L(j)} = z|_{L(j')}$ for some pair of distinct integers j and j' (suppose $j < j'$). Let $L(j,j') = \bigcup_{i=j}^{j'-1} L(j)$. The word $A = z|_{L(j,j')}$ may be used to construct a periodic word. For $K = k+1$ and $J = j' - j$, the set $L(j,j')$ is a $K \times J$ rectangle of lattice points, so $\mathbb{Z}^2 = \bigcup_{(m,n) \in \mathbb{Z}^2} L(j,j') + (mK, nJ)$. If we define $x|_{L(j,j')+(mK,nJ)} = A$ it should be clear that x has a finite orbit and that our choice of A guarantees that x is locally X allowed. ∎

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