Morphisms from Non-periodic $\mathbb{Z}^2$ Subshifts II: Constructing Homomorphisms to Square Filling Mixing Shifts of Finite Type

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Contents

Abstract. Part II. Krieger's embedding theorem for $\mathbb{Z}$ mixing SFTs is extended to a $\mathbb{Z}^2$ case. We prove: if X is a non-periodic subshift ($\sigma^v x = x \ \Rightarrow v = 0 \in \mathbb{Z}^2$) and Y is a $\mathbb{Z}^2$ square filling mixing shift of finite type, then there exists a homomorphism $X \to Y$. The proof is a construction which begins by constructing Voronoi tilings using techniques from Part I. A tiling by Delaunay polygons is derived from the Voronoi tiling. The union of the boundaries of the Delaunay polygons (referred to as the Delaunay graph) is itself tiled by trees. Words painted on the thickened trees combine to form words on the thickened infinite Delaunay graph. It is the point and sole purpose of square filling that words on such thickened infinite graphs will correspond to points in the target space.

Combined with the results of Part I this gives us the $\mathbb{Z}^2$ extension of Krieger's Embedding Theorem: if X is a non-periodic subshift and Y is a $\mathbb{Z}^2$ square filling mixing shift of finite type, then there exists an embedding $X \hookrightarrow Y$ if and only if $h(X) < h(Y)$, where h denotes the $\mathbb{Z}^2$ entropy. The techniques developed here play a central role in the proof of an embedding theorem for general $\mathbb{Z}^2$ subshifts into square filling mixing shifts of finite type which will be carried out in a subsequent paper.

1. Introduction

In this paper we continue the program begun in Part I [L] of developing an extension of Krieger's embedding theorem to the $\mathbb{Z}^2$ case. For motivation see Part I. Here, we address the existence of homomorphisms from non-periodic $\mathbb{Z}^2$ subshifts into SFTs. A subshift X is non-periodic if for every $x \in X$, $\sigma^v x = x$ implies $v = 0$. As mentioned in Part I, there is a rich supply of non-periodic $\mathbb{Z}^2$ subshifts [Ro]. A $\mathbb{Z}^2$ matrix SFT Y is said to be square filling if for k and l suitably large, a locally Y allowed word u on the annulus $\Lambda_l \backslash \Lambda_{l-k}$ implies the existence of a Y allowed word w on Λ_l which agrees with u on $\Lambda_l \backslash \Lambda_{l-1}$. In this paper the primary result is, if X is a non-periodic $\mathbb{Z}^2$ subshift and Y is a square filling mixing SFT then there exists a homomorphism $\phi : X \to Y$.

Combined with the results of Part I we have the following $\mathbb{Z}^2$ extension of Krieger's embedding theorem. If X is a non-periodic $\mathbb{Z}^2$ subshift and Y is a square filling mixing SFT, then there is an embedding $X \hookrightarrow Y$ if and only if $h(X) < h(Y)$ where h is the $\mathbb{Z}^2$ entropy.

The primary result (Theorem 2.8) is not merely an academic exercise. The techniques used here are foundational and essential. They will be applied and extended in a subsequent paper where we prove for any $\mathbb{Z}^2$ subshift X and square filling mixing SFT Y, if there exist homomorphisms of the finite orbits of X into Y, then there exists a homomorphism of X into Y. This is a preliminary step to a $\mathbb{Z}^2$ extension of Krieger's embedding theorem which embeds arbitrary subshifts into square filling mixing SFTs given certain necessary conditions.

This paper is organized as follows. In Section 2 we present the necessary definitions and we present our results. The proof of the primary result consumes most of the paper. This proof is a construction that begins (in Section 3) by revisiting a technique detailed in [L] which constructs a clopen marking set that returns to itself regularly under the action of the shift. This set is used to construct a continuous shift commuting map $x \mapsto \mathcal{M}_x$ whose image is a "regular" subset of $\mathbb{Z}^2$. From each such subset $\mathcal{M}_x$ we derive a Voronoi tiling and then a Delaunay tiling.

Voronoi tilings and Delaunay tilings have a sizable literature (see [OBS], [A] and their references), however to the author's knowledge no one has addressed the content of Theorem 6.1 (specifically Parts 5 and 6). It is not that Theorem 6.1 is difficult, it merely seems to be a direction that has not been of much concern. To provide a proof of Theorem 6.1 we need an agreeable place to begin. Thus, in Section 4 we define a linear graph and gather together a few facts about Voronoi tiles which are essentially known and easily proven. In Section 5 we define Delaunay polygons and prove several relevant results about them, one of which allows us to conclude that there exists a continuous shift commuting map from any non-periodic subshift to the space of tilings of $\mathbb{R}^2$ with Delaunay polygons. In Section 6 we prove that the infinite linear graphs formed by the unions of the boundaries of the Delaunay tiles of such tilings of $\mathbb{R}^2$ may themselves be tiled by tree prototiles. More importantly, we show (for relevant circumstances) that the intersection of the neighborhoods of two distinct such tree tiles is simply a ball (or empty).

In Section 7 we exploit this tree tiling with "simple" neighborhood intersections to construct words on the tree-tile neighborhoods which agree on their intersections. We use these words to paint symbols on a neighborhood of the Delaunay tiling boundaries. It is the sole point and purpose of square filling that such words extend to points. How square filling guarantees this extension is detailed in Section 8. In Section 9 we combine the various results to construct our desired homomorphism and thus complete the proof of our primary result. In Section 10 we comment briefly on the possibilities of extension to $d > 2$ and in Section 11 we give some examples.

This paper is a consequence of a project which has been going on for several years during which I have accumulated several debts which I would like to acknowledge. I would like to thank Mike Boyle for many discussions and support during the time in which the ideas in this paper were being worked out and most recently for offering suggestions on an earlier version of this paper. I would like to thank Klaus Schmidt and the Erwin Schrödinger Institute for their warm hospitality and I would like to acknowledge the Fonds zur Förderung der wissenschaftlichen Forschung (FWF) for financial support during the year 1998-99 (research grant number P12250-MAT). Finally, I would like to thank the Pacific Institute of

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2. Definitions and Results.

For $x = (x_1, x_2) \in \mathbb{R}^2$ let $\|x\|$ refer to the Euclidean norm of x, and let $\|x\|_{\text{sup}} \equiv \max\{|x_1|, |x_2|\}$. For two points $x, y \in \mathbb{R}^2$ let $d(x,y) \equiv \|x-y\|$. We will regard $\mathbb{Z}^2$ as naturally embedded in $\mathbb{R}^2$ thus inheriting the above norms and distance function. For $r > 0$ and $v \in \mathbb{R}^2$ let $B(v,r) \equiv \{t \in \mathbb{R}^2 : d(v,t) < r\}$, and for $k \in \mathbb{N}$ and $v \in \mathbb{R}^2$ let $\Lambda_k + v \equiv \{t \in \mathbb{R}^2 : \|t-v\|_{\text{sup}} \leq k\}$ (writing $\Lambda_k \equiv \Lambda_k + 0$ and $\Lambda_k \backslash 0 \equiv \Lambda_k \backslash \{0\}$). For $A \subset \mathbb{R}^2$ we will let $\overline{A}$ denote the closure of A and ∂A denote the topological boundary of A with respect to the topology induced by the metric d. So $\overline{B}(v,r) = \{t \in \mathbb{R}^2 : d(v,t) \leq r\}$.

Let S be a finite set of symbols and let $S^{\mathbb{Z}^2}$ be the set of maps from $\mathbb{Z}^2$ to S with the product topology. The group $\mathbb{Z}^2$ acts naturally on $S^{\mathbb{Z}^2}$ and this action is referred to as the shift on $S^{\mathbb{Z}^2}$. Specifically, an element $g \in \mathbb{Z}^2$ takes $x \in S^{\mathbb{Z}^2}$ to the element $\sigma^g x \in S^{\mathbb{Z}^2}$ where $(\sigma^g x)_v = x_{g+v}$, $\forall v \in \mathbb{Z}^2$. The map σ^g acts homeomorphically on the set $S^{\mathbb{Z}^2}$ with respect to the product topology. A closed shift invariant subset $X \subset S^{\mathbb{Z}^2}$ is called a $\mathbb{Z}^2$ *subshift*. For X and Y subshifts, a continuous map $\phi : X \to Y$ which commutes with the shifts on X and Y is called a *homomorphism*. A surjective (injective, bijective) homomorphism is called a *factor map (embedding, topological conjugacy)*. A $\mathbb{Z}^2$ subshift X is *mixing* if for any pair of non-empty open sets U and V, we have $U \cap \sigma^g V \neq \emptyset$ for all but finitely many $g \in \mathbb{Z}^2$.

For $\mathcal{A} \subset \mathbb{Z}^2$ and $x \in S^{\mathbb{Z}^2}$ let $x|_{\mathcal{A}}$ be the element of $S^{\mathcal{A}}$ formed by the restriction of x to $\mathcal{A}$. For a $\mathbb{Z}^2$ subshift $X \subset S^{\mathbb{Z}^2}$ let $W_X(\mathcal{A}) \equiv \{w \in S^{\mathcal{A}} : \exists\, x \in X \text{ such that } x|_{\mathcal{A}} = w\}$ which we call the set of X *allowed* words on $\mathcal{A}$. If $\mathcal{B} \subset \mathbb{R}^2$ we let $x|_{\mathcal{B}} \equiv x|_{\mathcal{B} \cap \mathbb{Z}^2}$. Similarly, we will write $W_X(\mathcal{B}) \equiv W_X(\mathcal{B} \cap \mathbb{Z}^2)$. If the cardinality of $\mathcal{A} \subset \mathbb{Z}^2$ is finite (written $|\mathcal{A}| < \infty$) then we say $w \in W_X(\mathcal{A})$ is a *finite* word, otherwise we say w is an *infinite* word. The property of mixing may be expressed with finite words: for $\mathcal{A} \subset \mathbb{Z}^2$ finite and for $u, v \in W_X(\mathcal{A})$ and for all but finitely many $g \in \mathbb{Z}^2$ there exists $x \in X$ such that $x|_{\mathcal{A}} = u$ and $(\sigma^g x)|_{\mathcal{A}} = v$ (we say x *exhibits* u and v separated by g).

It will be useful to have a notion of a shift on words. For $\mathcal{A} \subset \mathbb{Z}^2$ and $v \in \mathbb{Z}^2$, define $\sigma^v(\mathcal{A}) \equiv \{a - v : a \in \mathcal{A}\} = \mathcal{A} - v$. For $w \in S^{\mathcal{A}}$ and $v \in \mathbb{Z}^2$, let us define $\sigma^v w \in S^{\sigma^v \mathcal{A}}$ as $(\sigma^v w)_u \equiv w_{u+v},\ \forall\, u \in \sigma^v \mathcal{A}$. This allows us to write $\sigma^v(x|_U) = (\sigma^v x)|_{\sigma^v U}$. If $\mathcal{A} \neq \sigma^v \mathcal{A}$, then $S^{\mathcal{A}}$ and $S^{\sigma^v \mathcal{A}}$ are distinct, which means that $W_X(\mathcal{A})$ and $W_X(\sigma^v \mathcal{A})$ are distinct, nonetheless when $\mathcal{A}$ is finite we will not hesitate to identify their elements.

Definition 2.1. A $\mathbb{Z}^2$ subshift $Y \subset S^{\mathbb{Z}^2}$ is a *shift of finite type* (SFT) if there exists an $n > 0$ such that

$$Y = \{x \in S^{\mathbb{Z}^2} : x|_{\Lambda_n + v} \in W_Y(\Lambda_n)\ \forall\, v \in \mathbb{Z}^2\}. \tag{1}$$

We say Y is Λ_n *scaled.*

Remark 2.2. The property of being a $\mathbb{Z}^2$ SFT is preserved under conjugacy but its scale is not.

If Equation 1 holds with some other $\mathcal{A} \subset \mathbb{Z}^2$ substituted for Λ_n, then we say Y is $\mathcal{A}$ scaled. For example we will speak of an SFT as being $[1,k]^2$ *scaled.* We say $Y \subset S^{\mathbb{Z}^2}$ is a $\mathbb{Z}^2$ *matrix* SFT if for some pair A and B of $|S| \times |S|$, 0-1 matrices we have $Y = \{y \in S^{\mathbb{Z}^2} : A(y_v, y_{v+e_1}) = 1$ and $B(y_v, y_{v+e_2}) = 1\ \forall\, v \in \mathbb{Z}^2\}$ where e_1 and e_2 are the standard unit vectors generating $\mathbb{Z}^2$. It is well known that any SFT is conjugate to a matrix SFT.

Definition 2.3. For an SFT $Y \subset S^{\mathbb{Z}^2}$ and subsets $\mathcal{B}, \mathcal{C} \subset \mathbb{Z}^2$ (with $\mathcal{B} + u \subset \mathcal{C}$ for some $u \in \mathbb{Z}^2$) let

$$W_Y^{loc(\mathcal{B})}(\mathcal{C}) \equiv \{w \in S^{\mathcal{C}} : w|_{\mathcal{B}+v} \in W_Y(\mathcal{B} + v) \ \text{ whenever } \mathcal{B} + v \subset \mathcal{C}\}$$

be the set of $\mathcal{B}$ *locally* Y *allowed words on* $\mathcal{C}$.

In a typical application Y is a $[1,k]^2$ scaled SFT, $2N + 1 > k$, and we work with $W_Y^{loc([1,k]^2)}(\Lambda_N)$. It is one of the hallmarks of $\mathbb{Z}^2$ SFTs that in general for an Λ_k scaled SFT Y and $(N > k)$ we have $W_Y(\Lambda_N) \subsetneq W_Y^{loc(\Lambda_k)}(\Lambda_N)$. Determining when an element of the larger is found in the smaller is in general not decidable [B]. If Y is a matrix SFT and

$\mathcal{C} \subset \mathbb{Z}^2$, we will let $W_Y^{loc}(\mathcal{C})$ denote the collection of words on $\mathcal{C}$ which are locally allowed according to the matrices.

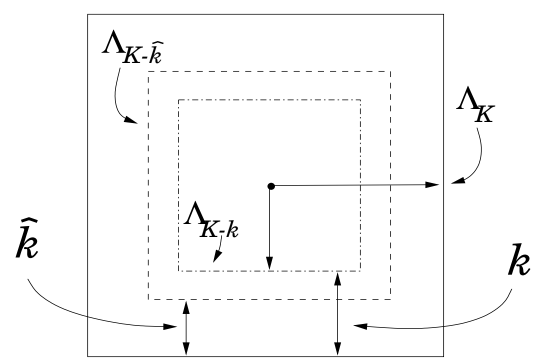
We now formulate a stronger mixing-like condition for $\mathbb{Z}^2$ subshifts which incorporates the geometry of $\mathbb{Z}^2$. See Figure 1.

Definition 2.4. For $0 < \widehat{k} \leq k \leq l$, a SFT Y is $(\widehat{k}, k, l)$ *square filling* if Y is $[1, \widehat{k}]^2$ scaled and whenever $K \geq l$ and $u \in W_Y^{loc([1,\widehat{k}]^2)}(\Lambda_K \backslash \Lambda_{K-k})$ there exists $w \in W_Y^{loc([1,\widehat{k}]^2)}(\Lambda_K)$ such that $u|_{\Lambda_K \backslash \Lambda_{K-\widehat{k}}} = w|_{\Lambda_K \backslash \Lambda_{K-\widehat{k}}}$. We say an SFT Y is *square filling (SF)* if Y is $(\widehat{k}, k, l)$ square filling for some $\widehat{k}, k$ and l.

The complexity of this definition has two sources. First, square filling is intended to be a local construction tool and the presence of the parameter $\widehat{k}$ ensures this. This is demonstrated in Section 11, Example 2. Second, it is intended that the following remark should hold.

Remark 2.5. The property of being a square filling SFT is preserved under conjugacy.

FIGURE 1. Square Filling.



A compelling (though less precise) restatement of the definition of square filling is, if u is a locally allowed word on a square annulus which can be filled in part way, then it can be filled in all the way. If Y is a $\mathbb{Z}^2$ matrix SFT, then in the definition we need only require $1 \leq k \leq l$, $u \in W_Y^{loc}(\Lambda_K \backslash \Lambda_{K-k})$, $w \in W_Y^{loc}(\Lambda_K)$ and that u and w agree on $\Lambda_K \backslash \Lambda_{K-1}$. In this case, we refer to Y as a (k, l) square filling matrix SFT.

Definition 2.6. A subshift X such that $x \neq \sigma^p x$ for every $x \in X$ and $p \in \mathbb{Z}^2 \backslash 0$ is called *non-periodic* since it has no periodic points.

Remark 2.7. A nonempty $\mathbb{Z}^2$ SFT is non-periodic if and only if it has no finite orbits. This is in contrast to $\mathbb{Z}$ subshifts where a $\mathbb{Z}$ SFT is empty if and only if it has no finite orbits.

Non-periodic subshifts are plentiful. Indeed, suppose X and Y are subshifts and let $Z = X \times Y$ where for $z = (x, y)$ we let $\sigma^v z = (\sigma^v x, \sigma^v y)$. Since $\sigma^v z = z$ if and only if both $\sigma^v x = x$ and $\sigma^v y = y$, the product space Z is non-periodic if either X or Y is non-periodic. Since entropy is additive under the formation of products, the existence of zero entropy non-periodic subshifts [M, Ra] implies the existence non-periodic subshifts of every entropy. With the embedding result (Theorem 2.13) we can conclude that every square filling mixing SFT contains an uncountable number of non-periodic subshifts.

In this paper we will prove the following via a construction.

Theorem 2.8. *If X is a non-periodic $\mathbb{Z}^2$ subshift and if Z is a $\mathbb{Z}^2$ square filling mixing SFT, then there exists a homomorphism $X \to Z$.*

Closely related to square filling and much easier to work with is the following.

Definition 2.9. A $\mathbb{Z}^2$ subshift Y is *square mixing* (SM) if there exists $\widetilde{k}$ such that whenever $k \geq \widetilde{k}$, $n \geq 0$ and $x, y \in Y$ then there exists $z \in Y$ such that $z|_{\Lambda_n} = x|_{\Lambda_n}$ and $z|_{\mathbb{Z}^2 \backslash \Lambda_{n+k}} = y|_{\mathbb{Z}^2 \backslash \Lambda_{n+k}}$. We refer to $\widetilde{k}$ as the square mixing parameter.

Remark 2.10. The homomorphic image of a square mixing subshift is square mixing.

Lemma 2.11. *Let Y be a $\mathbb{Z}^2$ mixing SFT. If Y is square filling, then Y is square mixing.*

In Section 8 we develop a technique for square filling mixing SFTs called stitching. Since stitching provides a nice way to prove Lemma 2.11, we will defer the proof of it until then. The converse of Lemma 2.11 is not true, which we demonstrate with an example in Section 11. In Part I [L] we proved the following.

Theorem 2.12. *Let X be a non-periodic $\mathbb{Z}^d$ subshift, let Z be a $\mathbb{Z}^d$ square mixing SFT and suppose there exists a homomorphism $\phi : X \to Z$. Then there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$, where h denotes the $\mathbb{Z}^d$ entropy.*

Together these results give us the following.

Theorem 2.13. *Let X be an non-periodic $\mathbb{Z}^2$ subshift and let Z be a $\mathbb{Z}^2$ square filling mixing SFT. There exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

We finish this section with an outline of the proof of Theorem 2.8. We begin (in Section 3) by reviewing a technique detailed in [L]. For $m \geq 1$, we define $\mathfrak{M}_m$ as the collection of all subsets $\mathcal{M} \subset \mathbb{Z}^2$ with the properties that 1) $\overline{B}(v,m) \cap \mathcal{M} = \{v\}$ for all $v \in \mathcal{M}$ and 2) for every $v \in \mathbb{Z}^2$, $\overline{B}(v,m) \cap \mathcal{M} \neq \emptyset$. The technique constructs a clopen marking set F that returns to itself "regularly" under the action of the shift. This set is used to construct a continuous shift commuting map $x \mapsto \mathcal{M}_x \equiv \{v \in \mathbb{Z}^2 : \sigma^v x \in F\} \in \mathfrak{M}_m$.

From each subset $\mathcal{M} \in \mathfrak{M}_m$ we derive the Voronoi tiling $\{(V_v, v) : v \in \mathcal{M}\}$ where $V_v = \{r \in \mathbb{R}^2 : d(r,v) \leq d(r,\mathcal{M})\}$, and then we derive the Delaunay tiling. For $p \in \mathbb{R}^2$ let $\mathcal{M}(p) = \{v \in \mathcal{M} : d(p,v) = d(p,\mathcal{M})\}$. The Delaunay tiling is the set $\{(\mathcal{H}(\mathcal{M}(p)), p) : |\mathcal{M}(p)| \geq 3\}$ where $\mathcal{H}(A)$ denotes the closed convex hull of a set $A \subset \mathbb{R}^2$. In order to provide a proof of the central result of Section 6 (Theorem 6.1) we collect together (in Sections 4 and 5) a few useful and easily proven results about Voronoi and Delaunay tilings, including the fact that (in these circumstances) both Voronoi and Delaunay prototiles are convex polygons.

In Section 6 we define the *Delaunay graph* to be the infinite linear graph formed by the union of the boundaries of the Delaunay tiles which cover $\mathbb{R}^2$. We introduce Delaunay edges, show they are "exterior chords," and show that the boundary of each Delaunay tile is the union of such edges. Recognizing Delaunay edges as exterior chords allows us to quickly conclude that Delaunay edges only intersect at their endpoints, if at all, and that disjoint Delaunay edges are uniformly (with respect to m) separated. We use the Delaunay edges to construct a finite set $\mathfrak{T}_m$ of tree prototiles which we may use to tile the Delaunay graphs (in a continuous and shift commuting manner). Moreover, we take advantage of the uniform separation of the Delaunay edges to show the intersection of the neighborhoods of two distinct such tree tiles is simply a ball (or empty).

In Section 7 we exploit these tree tilings (with the above nice neighborhood intersections) to construct words on the tile neighborhoods which agree on their intersections. Briefly, we pick a locally allowed word α and, for each tree prototile, use square mixing to “place” in a point $y \in Y$ a copy of α at each of the terminal vertices of the tree. (See Figure 8.) Restricting the resulting point to a neighborhood of the tree gives us a word. We use these words to paint symbols on a neighborhood of the Delaunay tiling boundaries. As mentioned, it is the sole point and purpose of square filling that such words extend to points.

In Section 8 we detail how square filling guarantees this extension. Briefly, if Y is a mixing matrix SFT and if Y is square filling, then Y is “filling” on more general shapes, in particular, convex polygons. That is, for suitably “large” convex polygons if a word is locally allowed on a neighborhood of the boundary of the polygon, then there exists an allowed word on the whole polygon which agrees with the original word near the boundary.

Finally, in Section 9 we choose appropriate values for a few parameters which allow the results of the previous Sections (3-8) to hold simultaneously. Then we formalize the definition of the desired homomorphism and show it has the desired properties. Roughly, we have tiled the plane with Delaunay tiles and painted locally allowed words on a neighborhood of the union of the boundaries of these Delaunay tiles. Thus, each Delaunay polygon is a convex polygon with locally allowed symbols painted on a neighborhood of its boundary. Square filling allows us to “fill the symbols in” giving us a word on the Delaunay polygon which agrees with the original symbols near the boundary. Together, the words on the polygons in the Delaunay tiling form a point in the image.

3. Markers and Voronoi Tilings.

In this section we review a sequence of definitions and results which were introduced and proven in Part I [L] (and the notation will differ slightly). A word $w \in W_X(\Lambda_M)$ is said to be j *periodic* if for every pair $v, v+j \in \Lambda_M$ we have $w_v = w_{v+j}$. For a $\mathbb{Z}^2$ subshift X and positive integers m and M with $0 < m < M$, let $\{w_i\}_{i=1}^{K}$ be an enumeration of the following set of words.

$$\{w_i\}_{i=1}^K \equiv \{w \in W_X(\Lambda_M) : w \text{ is not } j \text{ periodic for any } j \in \overline{B}_m \backslash 0\}, \tag{2}$$

where $\overline{B}_m = \overline{B}(0, m)$, the closed ball of radius m. Let $(w_i) \equiv \{x \in X : x|_{\Lambda_M} = w_i\}$.

Algorithm 3.1. *For $0 < m < M$ and an enumeration of words $\{w_i\}_{i=1}^K$, we construct the set $F(m, M) \subset X$ as follows. Let $F_1 = (w_1)$, for $1 < n \leq K$ let $F_n = F_{n-1} \cup ((w_n) \backslash \bigcup_{j \in \overline{B}_m} \sigma^j F_{n-1})$, and let $F(m, M) = F_K$.* •

Reindexing the words $\{w_i\}_{i=1}^K$ may produce a different set $F(m, M)$ but any such $F(m, M)$ will be adequate for us. We fix $F = F(m, M)$ and use it to construct for each $x \in X$ a set of *markers*

$$\mathcal{M}_x \equiv \{v \in \mathbb{Z}^2 : \sigma^v x \in F\}. \tag{3}$$

The map $x \mapsto \mathcal{M}_x$ for $x \in X$ is shift commuting ($\sigma^v \mathcal{M}_x = \mathcal{M}_{\sigma^v x}$) and continuous in the sense that given $k > 0$ there exists K such that whenever $x|_{\Lambda_K} = y|_{\Lambda_K}$ then $\mathcal{M}_x|_{\Lambda_k} = \mathcal{M}_y|_{\Lambda_k}$.

A set $\mathcal{M} \subset \mathbb{R}^2$ is $\overline{B}_m$ *separated* if for every $v \in \mathcal{M}$, $(\overline{B}_m + v) \cap \mathcal{M} = \{v\}$. A set $\mathcal{M} \subset \mathbb{R}^2$ is $\overline{B}_m$ *syndetic with respect to* $\mathbb{Z}^2$ if $v \in \mathbb{Z}^2$ implies $(\overline{B}_m + v) \cap \mathcal{M} \neq \emptyset$. A $\overline{B}_m$ separated set $\mathcal{M} \subset \mathbb{R}^2$ which is $\overline{B}_m$ syndetic with respect to $\mathbb{Z}^2$ will be called m *regular.* Let $\mathfrak{M}_m = \{\mathcal{M} \subset \mathbb{Z}^2 : \mathcal{M} \text{ is } m \text{ regular}\}$. If $\mathcal{M} \subset \mathbb{Z}^2$ is $\overline{B}_m$ syndetic with respect to $\mathbb{Z}^2$, then for every $v \in \mathbb{R}^2$, $(\overline{B}_{m+1} + v) \cap \mathcal{M} \neq \emptyset$.

Lemma 3.2. [L] *Given a non-periodic subshift X and an integer $m > 0$ there exists an integer $M > m$ such that for any $F = F(m, M)$ and each $x \in X$, the set $\mathcal{M}_x \in \mathfrak{M}_m$.*

By a polygon P we mean the closed convex hull of a finite set of points whose interior (P's) is nonempty. If $\mathfrak{V}$ is a collection of polygons we say $\mathcal{V} \subset \mathfrak{V} \times \mathbb{Z}^2$ is a *regular* $\mathfrak{V}$ *covering of* $\mathbb{R}^2$ if *i*) $\mathbb{R}^2 = \bigcup_{(V,v) \in \mathcal{V}} V + v$ and *ii*) for distinct $(V, v), (V', v') \in \mathcal{V}$, $V + v$ and $V' + v'$ have disjoint interiors. Regular $\mathfrak{V}$ coverings of $\mathbb{R}^2$ embody our notion of a tiling of $\mathbb{R}^2$ by polygonal sets. We refer to the elements of $\mathfrak{V}$ as prototiles and the sets $V + v$ for $(V, v) \in \mathcal{V}$ as tiles.

Let us define a shift σ on the elements of $\mathfrak{V} \times \mathbb{Z}^2$. For $\beta = (V, u) \in \mathfrak{V} \times \mathbb{Z}^2$ and $v \in \mathbb{Z}^2$, define $\sigma^v(\beta) = (V, u - v)$. For $\mathcal{V} \subset \mathfrak{V} \times \mathbb{Z}^2$ let us write $\sigma^v(\mathcal{V})$ for the set $\{\sigma^v(\beta) : \beta \in \mathcal{V}\}$. Clearly, $\mathcal{V}$ is a regular $\mathfrak{V}$ covering of $\mathbb{R}^2$ if and only if $\sigma^v(\mathcal{V})$ is.

We define a topology on collections of regular $\mathfrak{V}$ coverings in which whenever two regular $\mathfrak{V}$ coverings $\mathcal{V}_1$ and $\mathcal{V}_2$ agree on some finite set $\mathfrak{V} \times \Lambda_n$, then they lie in a common basis set. Such collections of regular $\mathfrak{V}$ coverings are subsets of $\mathcal{P}(\mathfrak{V} \times \mathbb{Z}^2)$, where $\mathcal{P}(\mathcal{A})$ denotes the collection of all subsets of $\mathcal{A}$. Also, when $\mathfrak{V}$ is finite (as it will be for us), $\mathfrak{V} \times \mathbb{Z}^2$ is a discrete metric space. For a discrete metric space A define the *Bounded Neighborhood* (BN) topology on $\mathcal{P}(A)$ as that topology generated by the basis of sets consisting of all $\mathcal{U}(\alpha, B) \equiv \{\beta \in \mathcal{P}(A) : \alpha \cap B = \beta \cap B\}$ where $B \subset A$ is finite and $\alpha \in \mathcal{P}(A)$. With respect to this topology, the shift map σ defined above is continuous.

For $v \in \mathcal{M} \subset \mathbb{R}^2$ the *Voronoi tile* corresponding to v with respect to $\mathcal{M}$ is the set $V_v \equiv \{r \in \mathbb{R}^2 : d(r, v) \leq d(r, \mathcal{M})\}$ (where $d(u, v) = \|u - v\|$ is the Euclidean metric). The set $\{V_v\}_{v \in \mathcal{M}}$ covers $\mathbb{R}^2$ and for $u \neq v$, the sets V_u and V_v have disjoint interiors. The following lemma summarizes a few facts about "Voronoi" tilings generated by sets $\mathcal{M} \in \mathfrak{M}_m$.

Lemma 3.3. [L] *For $m \geq 1$ fixed, the collection $\mathfrak{V}_m = \{V_v - v : v \in \mathcal{M}$ for some $\mathcal{M} \in \mathfrak{M}_m\}$ is a finite set of convex polygonal prototiles such that for each prototile $V \in \mathfrak{V}_m$, $\overline{B}(0, m/2) \subset V \subset \overline{B}(0, m+1)$. In addition, for any $\mathcal{M} \in \mathfrak{M}_m$ the set $\mathcal{V}_m(\mathcal{M}) = \{(V_v - v, v) : v \in \mathcal{M}\}$ is a well-defined regular $\mathfrak{V}_m$ covering of $\mathbb{R}^2$ with the following properties.*

1) For every $v \in \mathcal{M}$ there exists a unique $(V, u) \in \mathcal{V}_m(\mathcal{M})$ such that $u = v$ and $V + u$ is the Voronoi prototile corresponding to v with respect to $\mathcal{M}$.

2) The map $\mathcal{M} \mapsto \mathcal{V}_m(\mathcal{M})$ is (BN) continuous.

3) $\mathcal{V}_m(\sigma^v \mathcal{M}) = \sigma^v \mathcal{V}_m(\mathcal{M})$.

We refer to $\mathcal{V}_m(\mathcal{M})$ as the *Voronoi tiling of $\mathbb{R}^2$ generated by $\mathcal{M}$* and to the set $\mathfrak{V}_m$ as the set of *Voronoi prototiles.* We will follow the convention used here and denote sets of prototiles with gothic script, the associated tilings with caligraphic script and the tiles themselves with the usual script.

4. Linear Graphs and the Voronoi Graphs.

In this section we define linear graphs and Voronoi graphs and we present a few results about Voronoi graphs which will be needed in Sections 5 and 6.

4.1. **Linear Graphs.** For any pair $u, v \in \mathbb{R}^2$ let $[u, v]$ refer to the line segment between u and v. By a *linear graph* G in $\mathbb{R}^2$ we will mean a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$, the set of *vertices*, is a discrete subset of $\mathbb{R}^2$ and $\mathcal{E}$, the set of *edges*, is such that each edge is the line segment $[u, v]$ between two distinct vertices $u, v \in \mathcal{V}$ and such that two distinct edges only intersect at their endpoints, if at all. If the vertex set of a linear graph G (in $\mathbb{R}^2$) is finite, then we will call G a *finite* graph. If each pair of distinct vertices can be connected by a finite sequence of edges, then we will say G is a *connected* graph. We will call a connected linear graph a *tree* if it has no "loops". We will refer to a vertex which occurs in exactly one line segment as a *terminal* vertex. If $G = (\mathcal{V}, \mathcal{E})$ is a linear graph, then we will refer to the *image* of G as the union of the edges, a subset of $\mathbb{R}^2$.

We are not so much interested in linear graphs in any sophisticated graph theoretical sense as much as we are simply looking for a language. We have an interest in the images of some linear graphs in $\mathbb{R}^2$ and we need a language which effectively addresses certain subsets of these images. Namely, those subsets which are edges and vertices of the underlying graph. We will speak of the "edges" and "vertices" of the image to refer to the images of the edges and vertices of the underlying graph.

For $\mathcal{M} \in \mathfrak{M}_m$ and $v \in \mathcal{M}$, each Voronoi tile V_v is a convex polygon, thus V_v is the closed convex hull of its extreme points. Let $\mathcal{V}_v$ be this set of extreme points ($|\mathcal{V}_v| < \infty$) and let $\mathcal{E}_v$ be the set of line segments $[u, w]$ (for distinct $u, w \in \mathcal{V}_v$) which do not intersect the interior of V_v. Then $(\mathcal{V}_v, \mathcal{E}_v)$ is a linear graph whose image is ∂V_v. It is this graph which we shall regard as the graph underlying ∂V_v.

4.2. **The Voronoi Graphs.** For $\mathcal{M} \in \mathfrak{M}_m$, we define $G(\mathcal{M})$ as follows,

$$G(\mathcal{M}) \equiv \bigcup_{(V,v) \in \mathcal{V}_m(\mathcal{M})} \partial V + v.$$

The graph underlying $G(\mathcal{M})$ is obtained as follows. For each $p \in \mathbb{R}^2$ we define $\mathcal{M}(p) = \{v \in \mathcal{M} : d(p, v) = d(p, \mathcal{M})\}$ and let $r_p = d(p, \mathcal{M})$ (thus, $\mathcal{M}(p) \subset \partial B(p, r_p)$ and $\mathcal{M}(p) \cap$

$B(p, r_p) = \emptyset$) where d denotes the Euclidean metric. We let $\mathcal{V}(\mathcal{M}) = \{v \in \mathbb{R}^2 : |\mathcal{M}(v)| \geq 3\}$. For each distinct pair $u, v \in \mathcal{M}$ we let $o(u, v) = \{r \in \mathbb{R}^2 : \mathcal{M}(r) = \{u, v\}\}$ and we let $\mathcal{E}(\mathcal{M}) = \{\overline{o(u, v)} : u, v \in \mathcal{M}$ are distinct$\}$. It is not hard to show the following (see [OBS] for related results).

Proposition 4.1. *For $\mathcal{M} \in \mathfrak{M}_m$ the following hold.*

i) For each $e \in \mathcal{E}(\mathcal{M})$ there exist distinct $p, q \in \mathcal{V}(\mathcal{M})$ such that $e = [p, q]$.

ii) For each $e \in \mathcal{E}(\mathcal{M})$ there exists a unique pair $u, v \in \mathcal{M}$ such that $e = V_v \cap V_u$, and conversely if $u, v \in \mathcal{M}$ are such that $e = V_v \cap V_u$ is a line segment then $e \in \mathcal{E}(G(\mathcal{M}))$.

iii) For each $p \in \mathcal{V}(\mathcal{M})$ there exist $e \in \mathcal{E}(\mathcal{M})$ and $q \in \mathcal{V}(\mathcal{M})$ such that $e = [p, q]$.

iv) For $e, f \in \mathcal{E}(\mathcal{M})$, $e \cap f \neq \emptyset$ if and only if $e = [p, q]$ and $f = [q, r]$ for some $p, q, r \in \mathcal{V}(\mathcal{M})$. If $e \neq f$ and $e \cap f \neq \emptyset$, then $e \cap f$ is a single point.

v) $G(\mathcal{M}) = \bigcup\{e : e \in \mathcal{E}(\mathcal{M})\}$.

In short, the pair $(\mathcal{V}(\mathcal{M}), \mathcal{E}(\mathcal{M}))$ is a linear graph which we refer to as the *Voronoi graph associated to* $\mathcal{M}$, though often we will just refer to $G(\mathcal{M})$ as the Voronoi graph. Because $\mathcal{V}_m(\mathcal{M}) = \{(V_v - v, v) : v \in \mathcal{M}\}$, we have $G(\mathcal{M}) = \bigcup_{v \in \mathcal{M}} \partial V_v$ where each V_v is the Voronoi tile corresponding to v with respect to $\mathcal{M}$. One can show the set of vertices (edges) of the graph underlying $G(\mathcal{M})$ equals the union (over $v \in \mathcal{M}$) of the sets of vertices (edges) of the graphs underlying the ∂V_v. Conversely, the vertices and edges in the graph underlying each ∂V_v are those vertices and edges in the graph underlying $G(\mathcal{M})$ whose images are subsets of V_v.

If $\mathcal{N}$ is a discrete subset of the circle S^1, then for any two distinct points $x, y \in \mathcal{N}$ there exist two arcs $A_1, A_2 \subset S^1$ such that $A_1 \cap A_2 = \{x, y\}$. If $A_1 \cap \mathcal{N} = \{x, y\}$ or $A_2 \cap \mathcal{N} = \{x, y\}$, then we say x and y are *adjacent* in S^1 with respect to $\mathcal{N}$. The following is elementary to prove and we leave it to the reader.

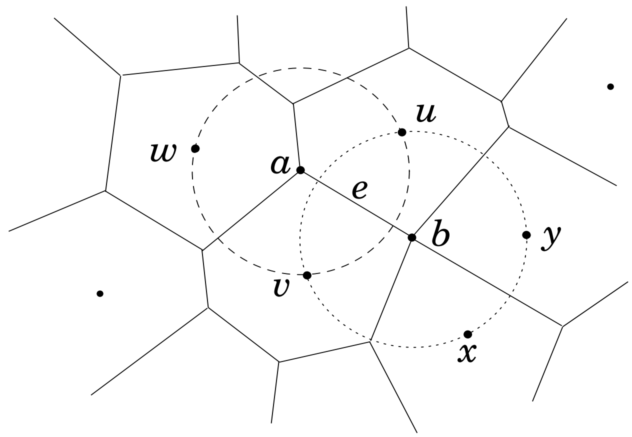


Figure 2. Each pair of distinct elements from $\{u, v, w\} = \mathcal{M}(a)$ are adjacent in $\partial B(a, r_1)$ with respect to $\mathcal{M}(a)$. Whereas in $\mathcal{M}(b) = \{u, v, x, y\}$ only the pairs $(u, v), (v, x), (x, y), (u, y)$ are adjacent in $\partial B(b, r_b)$ with respect to $\mathcal{M}(b)$.

Proposition 4.2. *If $a \in \mathcal{V}(\mathcal{M})$ and $u, v \in \mathcal{M}(a)$ are adjacent in $\partial B(a, r_a)$ with respect to $\mathcal{M}(a)$, then there exists an edge $e = [a, b] \in \mathcal{E}(\mathcal{M})$ such that $e = V_u \cap V_v$. And if $e = [a, b] \in \mathcal{E}(\mathcal{M})$, then $e = V_u \cap V_v$ where $u, v \in \mathcal{M}(a)$ are adjacent in $\partial B(a, r_a)$ with respect to $\mathcal{M}(a)$.*

When an edge $e = V_u \cap V_v$ we say e is *labelled* by u and v, since no other pair of distinct elements from $\mathcal{M}$ can label e. Figure 2 presents a picture of a subset of a typical $G(\mathcal{M})$ in which the content of Proposition 4.2 is demonstrated.

5. Delaunay Tilings

In this section we define a new tiling, the Delaunay tiling, and prove a few facts about these tilings which are summarized in Lemma 5.1. For any finite set of points S let $\mathcal{H}(S)$ denote the closed convex hull of S. For each $p \in \mathcal{V}(\mathcal{M})$ let

$$D(p) = \mathcal{H}(\mathcal{M}(p))$$

which we refer to as the *Delaunay polygon associated to p*. Because $p \in \mathcal{V}(\mathcal{M})$ implies $3 \leq |\mathcal{M}(p)| < \infty$ and $\mathcal{M}(p) \subset \partial B(p, r_p)$ (where $r_p > 0$) it follows that $D(p)$ is a convex polygon (with a nonempty interior). We might also note $D(p)$ need not contain p. Let

$$\mathfrak{D}_m = \{D(p) - p : p \in \mathcal{V}(\mathcal{M}) \text{ for some } \mathcal{M} \in \mathfrak{M}_m\}$$

and let $\mathcal{D}_m(\mathcal{M}) = \{(D(p) - p, p) : p \in \mathcal{V}(\mathcal{M})\}$. We refer to $\mathfrak{D}_m$ as the set of *Delaunay prototiles* and to $\mathcal{D}_m(\mathcal{M})$ as the *Delaunay tiling generated by* $\mathcal{M} \in \mathfrak{M}_m$.

Lemma 5.1. *There exists $\kappa > 1$ such that for each $m \geq 1$ the following holds. The set $\mathfrak{D}_m$ is a finite set of convex polygonal prototiles such that each prototile $D \in \mathfrak{D}_m$ has $B(q, m/\kappa) \subset D \subset B(p, \kappa m)$ for some $q, p \in \mathbb{R}^2$. Moreover, for every $\mathcal{M} \in \mathfrak{M}_m$, the set $\mathcal{D}_m(\mathcal{M})$ is a well-defined regular $\mathfrak{D}_m$ covering of $\mathbb{R}^2$ with the following properties.*

1) For every $p \in \mathcal{V}(\mathcal{M})$ there exists a unique $(D, q) \in \mathcal{D}_m(\mathcal{M})$ such that $q = p$ and $D + q$ is the Delaunay polygon associated to p.

2) The map $\mathcal{M} \mapsto \mathcal{D}_m(\mathcal{M})$ is continuous.

3) $\mathcal{D}_m(\sigma^v \mathcal{M}) = \sigma^v \mathcal{D}_m(\mathcal{M})$.

Proof of 5.1. We begin by proving the finiteness of $|\mathfrak{D}_m|$. Given $\mathcal{M} \in \mathfrak{M}_m$ and $p \in \mathcal{V}(\mathcal{M})$, we have $\mathcal{M}(p) \subset \mathbb{Z}^2$ and $\mathcal{M}(p) \subset \partial B(p, r)$ for some $r \in [m/2, m+1]$ (by Lemma 3.3). By a suitable shift, we may assume $p \in [0,1]^2$ and hence $\mathcal{M}(p) \subset \overline{B}(0, m+3) \cap \mathbb{Z}^2$, of which, there are are only a finite number of configurations, hence there are only finitely many such $p \in [0,1]^2$ and $\mathcal{H}(\mathcal{M}(p))$ and thus $D(p) - p$. In sum $|\mathfrak{D}_m| < \infty$.

Let us note the following proposition, the proof of which we leave to the reader because of its elementary nature. When we say a set $\mathcal{V}$ is 1 *separated* we mean for each pair $u, v \in \mathcal{V}$ of distinct elements, $d(u, v) \geq 1$.

Proposition 5.2. *Let $1/2 \leq r \leq 3/2$. There exists $\alpha \in (0,1)$ such that if P is any (convex) polygon inscribed in $\partial B(0, r)$ whose vertex set is 1 separated, then $B(q, \alpha) \subset P$ for some $q \in P$.*

Claim 5.3. *There exists $\kappa > 1$ such that for every $m \geq 1$, $\mathcal{M} \in \mathfrak{M}_m$, and $p \in \mathcal{V}(\mathcal{M})$*

$$B(q, m/\kappa) \subset D(p) \subset B(p, \kappa m)$$

for some $q \in \mathbb{R}^2$.

Proof of 5.3. Since $D(p) = \mathcal{H}(\mathcal{M}(p))$ and since $\mathcal{M}(p) \subset \partial B(p, r_p)$, each $D(p)$ is a polygon circumscribed in $\partial B(p, r_p)$ where $m/2 \le r_p \le m+1$ and the elements $\mathcal{M}(p)$ are m separated. Thus, scaling Proposition 5.2 by a factor of m we have, that $B(q', m\alpha) \subset D(p)$ for some $q' \in D(p)$. Since $D(p)$ is circumscribed in $\partial B(p, r_p)$ for $m/2 \le r_p \le m+1$, it follows $D(p) \subset B(p, m+2)$. Letting $\kappa = \max\{1/\alpha, \frac{1+2}{1}\}$, the lemma follows. ■

The fact that the $D(p) : p \in \mathcal{V}(\mathcal{M})$ cover $\mathbb{R}^2$ regularly is well known and straight forward to prove (see [OBS] for a closely related result) and we will leave it to the reader. We now address Parts 1, 2 and 3.

1) For $p \in \mathcal{V}(\mathcal{M})$, by the definition of $\mathcal{D}_m(\mathcal{M})$, the only element $(D', q) \in \mathcal{D}_m(\mathcal{M})$ with $q = p$ is $D' = D(p) - p$. So $D' + q = D(p)$.

2) We use the notation $\overline{B}_n = \overline{B}(0, n)$ and $\mathcal{M}_n = \mathcal{M} \cap \overline{B}_n$. Given $n > 0$ and $\mathcal{M} \in \mathfrak{M}_m$, let $K = \max\{d(p, \mathcal{M}) : p \in \overline{B}_n\}$. Let $\mathcal{M}'$ be such that $\mathcal{M}'_{n+K} = \mathcal{M}_{n+K}$. Then for each $p \in \overline{B}_n$ we have $\mathcal{M}'(p) = \mathcal{M}(p)$. Thus $|\mathcal{M}'(p)| \ge 3 \iff |\mathcal{M}(p)| \ge 3$, implying $\mathcal{V}(\mathcal{M}')_n = \mathcal{V}(\mathcal{M})_n$. For each $p \in \mathcal{V}(\mathcal{M}')_n$ it also follows that $D'(p) = \mathcal{H}(\mathcal{M}'(p)) = \mathcal{H}(\mathcal{M}(p)) = D(p)$. Thus $\mathcal{D}_m(\mathcal{M}) \cap (\mathfrak{D}_m \times \overline{B}_n) = \mathcal{D}_m(\mathcal{M}') \cap (\mathfrak{D}_m \times \overline{B}_n)$ which implies $\mathcal{M} \mapsto \mathcal{D}_m(\mathcal{M})$ is continuous. For $\mathcal{M} \in \mathfrak{M}_m$, since K has a uniform upper bound, we have uniform continuity.

3) Shift commutativity. One can check that for $p \in \mathbb{R}^2$ that $\sigma^v(\mathcal{M}(p)) = (\sigma^v \mathcal{M})(\sigma^v p)$ from which it follows $\sigma^v \mathcal{V}(\mathcal{M}) = \mathcal{V}(\sigma^v \mathcal{M})$ and $\sigma^v D(p) = D(\sigma^v p)$ where $D(\sigma^v p) = \mathcal{H}((\sigma^v \mathcal{M})(\sigma^v p))$. Applying these to the definitions of $\sigma^v \mathcal{D}_m(\mathcal{M})$ and $\mathcal{D}_m(\sigma^v \mathcal{M})$ one obtains equality. ■

6. Tiling Delaunay Graphs.

The purpose of this section is to prove Theorem 6.1, the statement of which requires a little preparation. Let us say a line segment $[a, b]$ is nontrivial if $a \ne b$ (*i.e.* $[a, b]$ has positive length). Recall, by Lemma 3.3 Part 1, for each $v \in \mathcal{M} \in \mathfrak{M}_m$ there exists a unique $(V, v) \in \mathcal{V}_m(\mathcal{M})$. For $\mathcal{M} \in \mathfrak{M}_m$ and distinct $u, v \in \mathcal{M}$ let us call the line segment $[u, v]$ a *Delaunay edge*, writing $DE(u, v) = [u, v]$, if the intersection $V + v \cap U + u$ is a nontrivial line segment, where $(V, v), (U, u) \in \mathcal{V}_m(\mathcal{M})$. For later convenience let us extend the notation $DE(u, v)$ to any pair $u, v \in \mathcal{M}$ by adopting the convention that if $V + v \cap U + u$ is not a nontrivial line segment then $DE(u, v) = \emptyset$. The term Delaunay edge will refer only to those nonempty $DE(u, v)$. Also for convenience, for $(V, v) \in \mathcal{V}_m(\mathcal{M})$, since each $V + v = V_v$, the

Voronoi tile corresponding to v with respect to $\mathcal{M}$, we will often simply write V_v instead of $V + v$.

For each distinct pair $u, v \in \mathcal{M}$ for which there is a Delaunay edge ($DE(u,v) \neq \emptyset$) let p_{uv} be the midpoint of $DE(u,v)$. We will refer to $[u, p_{uv}]$ and $[v, p_{uv}]$ as *Delaunay half-edges*. For each $\mathcal{M} \in \mathfrak{M}_m$ and $u \in \mathcal{M}$, define

$$T_u \equiv \bigcup_{DE(u,v)\neq\emptyset} [u, p_{uv}]$$

and let $\mathfrak{T}_m = \{T_u - u : u \in \mathcal{M} \in \mathfrak{M}_m\}$. For $\mathcal{M} \in \mathfrak{M}_m$ let

$$\mathcal{T}_m(\mathcal{M}) \equiv \{(T_u - u, u) : u \in \mathcal{M}\}.$$

For each $\mathcal{M} \in \mathfrak{M}_m$ define the *Delaunay graph* $DG(\mathcal{M})$ associated with $\mathcal{M}$ as follows

$$DG(\mathcal{M}) \equiv \bigcup_{(D,v)\in\mathcal{D}_m(\mathcal{M})} \partial D + v.$$

Since $|\mathfrak{D}_m| < \infty$ and since the map $\mathcal{M} \mapsto \mathcal{D}_m(\mathcal{M})$ is continuous and shift commuting so is the map $\mathcal{M} \mapsto DG(\mathcal{M})$. For any subset $A \subset \mathbb{R}^2$ and $K \geq 0$ recall $A_K := \{r \in \mathbb{R}^2 : d(r, A) \leq K\}$.

Theorem 6.1. *For $m \geq 2$ and $\mathcal{M} \in \mathfrak{M}_m$ the following hold.*

1) Each $T \in \mathfrak{T}_m$ is a tree.

2) $DG(\mathcal{M}) = \bigcup\{T + d : (T,d) \in \mathcal{T}_m(\mathcal{M})\}$ is the image of a linear graph.

3) For distinct $(T,d), (T',d') \in \mathcal{T}_m(\mathcal{M})$ the intersection $(T+d) \cap (T'+d')$ is either, empty or a single point p, and in the latter case p is

i) a terminal vertex of both $T+d$ and $T'+d'$, and

ii) the midpoint of an edge in the graph of $DG(\mathcal{M})$.

4) The map $\mathcal{M} \to \mathcal{T}_m(\mathcal{M})$ is BN continuous and shift commuting.

Given $K > 0$ there exists m_K such that for $m \geq m_K$ and $\mathcal{M} \in \mathfrak{M}_m$ the following hold.

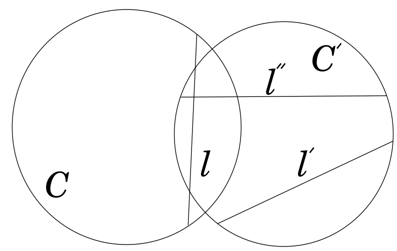
5) For $T \in \mathfrak{T}_m$, if p, p' are distinct terminal vertices of T, then $\overline{B}(p,K) \cap \overline{B}(p',K) = \emptyset$.

6) For every $(T,d), (T',d') \in \mathcal{T}_m(\mathcal{M})$ distinct, the intersection $(T+d)_K \cap (T'+d')_K$ is either empty or is $\overline{B}(p,K)$ where $p = (T+d) \cap (T'+d')$.

The proof of 6.1 is quite elementary, nonetheless we feel it should be included. Before we begin the proof, a brief digression is in order.

6.1. **Exterior Chords.** By a circle C we mean $\partial B(p, r)$ for some $p \in \mathbb{R}^2$ and $r > 0$. By the interior B of C we mean $B(p, r)$. A *chord* in C is a line segment whose endpoints both lie in C. By a chord in C *exterior* to the circle C' we mean a chord in C whose endpoints lie in $C \backslash B'$ where B' is the interior of C'. See Figure 3. By the *interior* of a chord we mean the chord minus its endpoints. The reader may verify the following two propositions.

Figure 3. The chord l in C is exterior to C' (and l' is a chord in C' exterior to C), but l'' is a chord in C' which is not exterior to C.



Proposition 6.2. *For any two distinct circles C and C' suppose l is a chord in C exterior to C' and suppose l' is a chord in C' exterior to C. If $l \neq l'$ then l and the interior of l' are disjoint, as are l' and the interior of l.*

We say a circle C has radius r if $C = \partial B(p, r)$ for some $p \in \mathbb{R}^2$. For a line segment l let $V(l)$ denote its endpoints and let $|l|$ denote its length.

Proposition 6.3. *There exists a $K' > 1$ such that if C and C' are distinct circles whose radii lie in the interval $[\frac{1}{2}, \frac{3}{2}]$ and if l and l' are any two chords such that l is a chord in C exterior to C', l' is a chord in C' exterior to C, and $|l|, |l'| \geq 1$, then $d(l, l') \geq d(V(l), V(l'))/K'$.*

Claim 6.4. *For $\mathcal{M} \in \mathfrak{M}_m$, each Delaunay edge l is a chord in a circle C with radius $r \in [\frac{m}{2}, m+1]$. Moreover, for any pair of distinct Delaunay edges, l and l', there are distinct circles C and C' (with radii $r, r' \in [m/2, m+1]$) such that l is a chord in C exterior to C' and l' a chord in C' exterior to C.*

Proof of 6.4. (See Figure 4.) Since $l = DE(u, v)$ is a Delaunay edge, $V_u \cap V_v$ is a nontrivial line segment; call it e. By Proposition 4.1 Part *ii*, $e \in \mathcal{E}(\mathcal{M})$ and (by Part *i*) $e = [a, b]$ where $a, b \in \mathcal{V}(\mathcal{M})$ are distinct. Now $a, b \in V_u \cap V_v$ and hence $u, v \in \mathcal{M}(a)$ and $u, v \in \mathcal{M}(b)$, so $DE(u, v) = [u, v]$ is a chord in both $\partial B(a, r_a)$ and $\partial B(b, r_b)$. Similarly $l' = DE(u', v')$ is a chord in $\partial B(a', r_{a'})$ and $\partial B(b', r_{b'})$ for similar $a', b' \in \mathcal{V}(\mathcal{M})$.

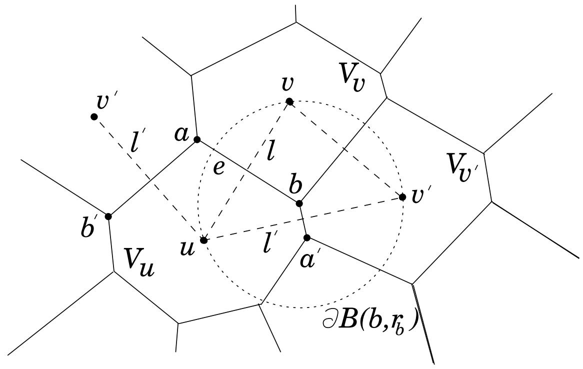


Figure 4. Voronoi Tiles, Delaunay Edges and a Delaunay Tile

Since $DE(u,v) \neq DE(u',v')$ we must have at least three of u, v, u', and v' are distinct (we already know $u \neq v$ and $u' \neq v'$); suppose u, v, and v' are distinct and $u = u'$. So, $V_v \cap V_u \cap V_{v'}$ can be at most a point (this being the nature of three distinct Voronoi tiles in the plane). Since $a' \neq b'$, $[a',b'] = V_{v'} \cap V_{u'}$, and since $[a,b] \cap [a',b'] \subset V_v \cap V_u \cap V_{v'}$, this implies $[a,b] \neq [a',b']$ and that $\{a,b\}$ and $\{a',b'\}$ are not equal as sets. Thus there exist elements, one from each set, which are distinct. Suppose $a \neq a'$. Then $\partial B(a, r_a)$ and $\partial B(a', r_{a'})$ are distinct circles. We get a similar result if $u \neq u'$.

Recall $\mathcal{M} \cap B(a, r_a) = \emptyset$ for all $a \in \mathbb{R}^2$, which implies the endpoints of $DE(u,v)$ do not lie inside $B(a', r_{a'})$ (or $B(b', r_{b'})$). That is, $DE(u,v)$ is a chord of $\partial B(a, r_a)$ exterior to $\partial B(a', r_{a'})$ and likewise $DE(u',v')$ is a chord of $\partial B(a', r_{a'})$ exterior to $\partial B(a, r_a)$.

That r and r' lie in $[m/2, m+1]$ is a consequence of Lemma 3.3. ■

Corollary 6.4.1. *For $\mathcal{M} \in \mathfrak{M}_m$, any pair of distinct Delaunay edges $l = DE(u,v)$ and $l' = DE(u',v')$ can only intersect at their endpoints, if at all.*

Proof of 6.4.1. By Claim 6.4, l is a chord in C exterior to C' and l' is a chord in C' exterior to C (for some distinct C and C'). The corollary follows from Proposition 6.2. ■

Corollary 6.4.2. *Given $K > 0$ there exists m_K such that for $m \geq m_K$, $\mathcal{M} \in \mathfrak{M}_m$ and any pair of Delaunay edges $l = DE(u,v)$ and $l' = DE(u',v')$, either $d(l,l') = 0$ or $d(l,l') \geq 2K$.*

Proof of 6.4.2. Let $m_K \geq 2K \cdot K'$ where K' is from Proposition 6.3. By Claim 6.4, for $l \neq l'$, l is a chord in C exterior to C' and l' is a chord in C' exterior to C (for some distinct C and C') where the radii of C and C' are in $[\frac{m}{2}, \frac{3m}{2}]$. Since $V(l), V(l') \subset \mathcal{M}$ we have $|l|, |l'| \geq$

m. It follows by Proposition 6.3 (after scaling by m) that $d(l,l') \geq d(V(l),V(l'))/K'$. Since $V(l),V(l') \subset \mathcal{M}$ it also follows that if $d(V(l),V(l')) > 0$, then $d(V(l),V(l')) \geq m$. Thus, if $d(l,l') > 0$ then $d(l,l') \geq d(V(l),V(l'))/K' \geq m/K' \geq 2K$. ■

Finally, before we begin the proof of Theorem 6.1 we have a few results addressing the relationship between Delaunay polygons and Delaunay edges.

Claim 6.5. *For $\mathcal{M} \in \mathfrak{M}_m$ and $p \in \mathcal{V}(\mathcal{M})$,*

$$\partial D(p) = \cup\{DE(u,v) : u,v \in \mathcal{M}(p) \text{ are adjacent in } \partial B(p,r_p) \text{ with respect to } \mathcal{M}(p)\}.$$

Proof of 6.5. Recall, for each $p \in \mathbb{R}^2$, $r_p = d(p,\mathcal{M})$, and $\mathcal{M}(p) = \mathcal{M} \cap \partial B(p,r_p)$. For $p \in \mathcal{V}(\mathcal{M})$, $|\mathcal{M}(p)| \geq 3$ and by Proposition 4.2, for each pair $u,v \in \mathcal{M}(p)$ adjacent in $\partial B(p,r_p)$ with respect to $\mathcal{M}(p)$ there is an edge in $e \in \mathcal{E}(\mathcal{M})$ such that $e = V_u \cap V_v$, and thus there is a Delaunay edge $DE(u,v)$ connecting u and v.

That is, we have a finite subset of a circle ($\mathcal{M}(p)$) and for each pair of elements ($u,v \in \mathcal{M}(p)$) which are adjacent with respect to this subset, there is a line segment ($DE(u,v)$) connecting the pair. Thus, these line segments bound a closed convex set $\mathcal{P}$. Because $|\mathcal{M}(p)| \geq 3$ there are at least 3 adjacent pairs from $\mathcal{M}(p)$ and thus the boundary of $\mathcal{P}$ contains at least three distinct line segments, hence each $\mathcal{P}$ has a nonempty interior and so $\mathcal{P}$ is in fact a closed convex polygon.

Since $\mathcal{P}$ is a closed convex polygon, it is the closed convex hull of its extreme points and by the nature of the construction of $\mathcal{P}$ those extreme points are $\mathcal{M}(p)$, *i.e.* $\mathcal{P} = \mathcal{H}(\mathcal{M}(p)) = D(p)$. We conclude $D(p)$ is a closed convex polygon whose boundary is the union we claimed. ■

In fact, we have more.

Claim 6.6. *For $u,v \in \mathcal{M}(p)$, $DE(u,v) \neq \emptyset$ if and only if u and v are adjacent in $\partial B(p,r_p)$ with respect to $\mathcal{M}(p)$. So $\partial D(p) = \cup\{DE(u,v) : u,v \in \mathcal{M}(p)\}$.*

Proof of 6.6. If $u,v \in \mathcal{M}(p)$ are such that $DE(u,v) \neq \emptyset$ then $e = V_u \cap V_v$ is a non-trivial line segment. That is, by Proposition 4.1(ii), $e \in \mathcal{E}(\mathcal{M})$, and by Proposition 4.2 u and v are adjacent in $\partial B(p,r_p)$. So, if u,v in $\mathcal{M}(p)$ are not adjacent in $\partial B(p,r_p)$, then $DE(u,v) = \emptyset$ and $\partial D(p) = \{DE(u,v) : u,v \in \mathcal{M}(p)\}$ follows. ■

Recalling that the "interior" of a line segment $[a, b]$ refers to $[a, b]\backslash\{a, b\}$, Claim 6.6 allows us to prove the following.

Claim 6.7. *The "interior" of any Delaunay edge and the topological interior of any Delaunay polygon are disjoint.*

Proof of 6.7. Since $D(p) \subset \overline{B}(p, r_p)$, the interior of $D(p)$ is a subset of $B(p, r_p)$ and since $B(p, r_p) \cap \mathcal{M} = \emptyset$, neither endpoint of any Delaunay edge occurs in the interior of any Delaunay polygon. Thus, any Delaunay edge which intersects a Delaunay polygon $D(p)$ must intersect the boundary of $D(p)$. Since, distinct Delaunay edges intersect (at most) at their endpoints and since $\partial D(p) = \cup\{DE(u, v) : u, v \in \mathcal{M}(p)\}$, any Delaunay edge l whose "interior" intersects $D(p)$ must have $V(l) \subset \{V(DE(u, v)) : u, v \in \mathcal{M}(p)\}$. But this implies $l = DE(u, v)$ for some $u, v \in \mathcal{M}(p)$ and thus by Claim 6.6, that $l \subset \partial D(p)$. Thus, the "interior" of l does not intersect the interior of $D(p)$. ■

This implies that distinct Delaunay polygons have disjoint interiors.

Proof of 6.1. Part 1). Each T_u is the union of a set $\{l_i\}_{i=1}^{L}$ of line segments for which $l_i = [u, p_i] \subset DE(u, v_i)$, where p_i is the midpoint of $DE(u, v_i)$. By Corollary 6.4.1 we know each pair of distinct Delaunay edges only intersects at their endpoints, thus for $i \neq j$, $l_i \cap l_j = u$. It follows, that if we let $V_u = \{u\} \cup \{p_i\}_{i=1}^{L}$ and let $E_u = \{l_i\}_{i=1}^{L}$, then the pair (V_u, E_u) is a linear graph whose image is T_u and this graph is a tree.

Part 2). We now show

$$DG(\mathcal{M}) = \bigcup_{u,v \in \mathcal{M}} DE(u, v). \tag{4}$$

By definition, $DG(\mathcal{M})$ is the union over $p \in \mathcal{V}(\mathcal{M})$ of $\partial D(p)$. So Claim 6.5 allows us to conclude the left hand side (LHS) is a subset of the right hand side (RHS). On the other hand, for each Delaunay edge $DE(u, v)$ there is an edge $e = [p, q] \subset V_u \cap V_v$ with $p, q \in \mathcal{V}(\mathcal{M})$. Thus $u, v \in \mathcal{M}(p)$, $DE(u, v) \subset \mathcal{H}(\mathcal{M}(p)) = D(p)$ and thus (by 6.7) $DE(u, v) \subset \partial D(p)$. Hence, the RHS is a subset of the LHS, giving us Equation 4.

We now prove the equality in Part 2 of 6.1. Because each Delaunay half edge $h \in T_u$ is a subset of some $DE(u, v)$ for $u, v \in \mathcal{M}$, the RHS is a subset of the LHS. Conversely, each $DE(u, v) \subset T_u \cup T_v$ hence the LHS is a subset of the RHS, and we have equality.

To check that $DG(\mathcal{M})$ is the image of a linear graph, let $\mathcal{V} = \mathcal{M}$ and let $\mathcal{E}$ be the set $\{DE(u,v) : u, v \in \mathcal{M}\}$. Because the $DE(u,v)$ only intersect (if at all) at their endpoints (Corollary 6.4.1), it follows that the pair $(\mathcal{V}, \mathcal{E})$ is a linear graph whose image (by Equation 4) is $DG(\mathcal{M})$.

Part 3). We check that for distinct $(T,v), (T',v') \in \mathcal{T}_m(\mathcal{M})$ that $T + v \cap T' + v'$ consists of at most one point. Suppose (T,v) and (T',v') are distinct, then $v, v' \in \mathcal{M}$ are distinct. Suppose $h = [v, p_{vs}]$ and $k = [v', p_{v't}]$ are Delaunay half edges with $v \neq v'$. Because distinct Delaunay edges only intersect at their endpoints (Corollary 6.4.1) and since $v \neq v'$, $h \cap k \neq \emptyset$ if and only if $v = t$ and $v' = s$. That is h and k intersect if and only if they are both subsets of the same Delaunay edge. Since T_v and $T_{v'}$ are unions of such half edges, T_v and $T_{v'}$ intersect if and only if $[v, v']$ is a Delaunay edge, in which case $T_v \cap T_{v'} = p_{vv'}$, which is both the midpoint of $D(v,v')$ and a terminal vertex for both T_v and $T_{v'}$. See Figure 5.

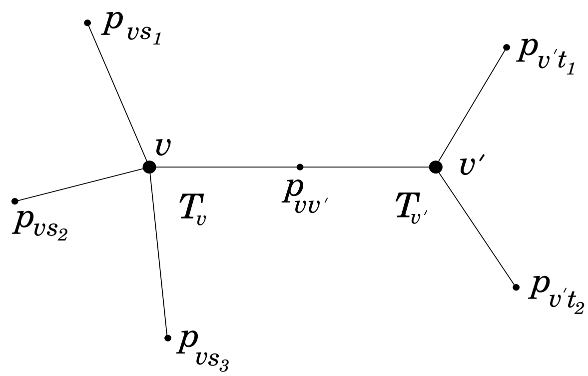


Figure 5. The Intersection of Trees

Part 4). To check continuity, observe a set of Delaunay polygons which covers $\overline{B}(0,n)$ determines the set of Delaunay edges which intersect $B(0,n)$ which in turn determine the T_v for $v \in \mathcal{M} \cap B(0,n)$. Since each $D \in \mathfrak{D}_m$ is circumscribed in $\partial B(0,r)$ for some $r \in [\frac{m}{2}, m+1]$ and since for any $\mathcal{M} \in \mathfrak{M}_m$, $\mathcal{D}_m(\mathcal{M})$ covers $\mathbb{R}^2$, the set

$$\{D + p : (D,p) \in \mathcal{D}_m(\mathcal{M}) \text{ and } p \in \overline{B}(0, n+m+1)\}$$

covers $\overline{B}(0,n)$. Since $\mathcal{M} \to \mathcal{D}_m(\mathcal{M})$ is continuous there exists a K such that if $\mathcal{M}' \in \mathfrak{M}_m$ agrees with $\mathcal{M}$ on $\overline{B}(0, n+K)$, then $\mathcal{D}_m(\mathcal{M})$ agrees with $\mathcal{D}_m(\mathcal{M}')$ on $\mathfrak{D}_m \times \overline{B}(0, n+m+1)$. Thus,

$\{DE(u,v) : u, v \in \mathcal{M} \text{ and } DE(u,v) \cap B(0,n) \neq \emptyset\} =$

$$\{DE'(u',v') : u',v' \in \mathcal{M}' \text{ and } DE'(u',v') \cap B(0,n) \neq \emptyset\}$$

implying $\{(T_u - u, u) : u \in \mathcal{M} \cap B(0,n)\} = \{(T'_{u'} - u', u') : u' \in \mathcal{M}' \cap B(0,n)\}$. That is, $\mathcal{M} \mapsto \mathcal{T}_m(\mathcal{M})$ is continuous.

To check the shift commutativity observe, because $\sigma^w \mathcal{V}_m(\mathcal{M}) = \mathcal{V}_m(\sigma^w \mathcal{M})$ (Lemma 3.3), for $u, v \in \mathcal{M}$, $DE(u,v)$ is a Delaunay edge if and only if $DE(\sigma^w u, \sigma^w v)$ is a Delaunay edge for $\sigma^w u, \sigma^w v \in \sigma^w \mathcal{M}$, and $\sigma^w DE(u,v) = DE(\sigma^w u, \sigma^w v)$. Thus for $\sigma^w u \in \sigma^w \mathcal{M}$,

$$T_{\sigma^w u} = \bigcup_{DE(\sigma^w u, \sigma^w v) \neq \emptyset} [\sigma^w u, p_{\sigma^w u, \sigma^w v}] = \bigcup_{DE(u,v) \neq \emptyset} \sigma^w [u, p_{uv}] = \sigma^w T_u.$$

So for $u \in \mathcal{M}$ and $\sigma^w u \in \sigma^w \mathcal{M}$, $T_v - v = T_{\sigma^w v} - \sigma^w v$, thus $\{T_u - u : u \in \mathcal{M}\} = \{T_{v'} - v' : v' \in \sigma^w \mathcal{M}\}$ and it follows that $\sigma^w \mathcal{T}_m(\mathcal{M}) = \mathcal{T}_m(\sigma^w \mathcal{M})$.

We skip Part 5) for the moment and address Part 6). Since each T_u is the union of the Delaunay half edges which intersect u we will be done if we can establish the next claim.

Claim 6.8. *Given $K > 0$ there exists m_K such that for $m \geq m_K$ and $\mathcal{M} \in \mathfrak{M}_m$ the following holds. For $u, v, u', v' \in \mathcal{M}$, if $h = [u, p_{uv}]$ and $h' = [u', p_{u'v'}]$ are Delaunay half edges with $u \neq u'$, then either $[u, p_{uv}]_K \cap [u', p_{u'v'}]_K = \emptyset$ or $v = u'$, $v' = u$ and $[u, p_{uv}]_K \cap [u', p_{u'v'}]_K = \overline{B}(p_{uv}, K)$.*

To prove Claim 6.8 let us begin by agreeing that if a, b and c are three points in general position, then (since their closed convex hull is a triangle) $\angle(a,b,c)$ denotes the measure of the interior angle at b. We leave the proof of the following proposition to the reader.

Proposition 6.9. *There exists $\theta_0 \in (0, \pi)$ such that for any circle C of radius $r \in [\frac{1}{2}, \frac{3}{2}]$ and any three points $a, b, c \in C$ with $d(a,b), d(b,c), d(a,b) \geq 1$, then $\angle(a,b,c) \geq \theta_0$.*

If $k = [a,b]$ and $l = [b,c]$ are two line segments whose endpoints a, b and c are in general position, then let us define $\angle(k,l) = \angle(a,b,c)$.

Claim 6.10. *For $m \geq 2$, $\mathcal{M} \in \mathfrak{M}_m$, and $p \in \mathcal{V}(\mathcal{M})$ suppose k and l are distinct Delaunay edges which are subsets of $\partial D(p)$ and $k \cap l \neq \emptyset$. Then $\angle(l,k) \geq \theta_0$.*

Proof of 6.10. Since $k, l \subset \partial D(p)$ and $k \cap l \neq \emptyset$ we may write $k = DE(u,v) = [u,v]$ and $l = DE(v,w) = [v,w]$ for some $u, v, w \in \mathcal{M}(p)$. Because $\mathcal{M}(p) = \mathcal{M} \cap \partial B(p, r_p)$ for

$r_p \in [\frac{m}{2}, m+1]$ and $d(u,v), d(v,w), d(u,w) \geq m$, it follows from Proposition 6.9 (scaled by m) that $\angle(l,k) = \angle(u,v,w) \geq \theta_0$. ■

Claim 6.11. *For $m \geq 2$ and $\mathcal{M} \in \mathfrak{M}_m$, if k and l are any two distinct Delaunay edges such that $k \cap l \neq \emptyset$, then $\angle(k,l) \geq \theta_0$.*

Proof of 6.11. Because $l = DE(u,v) \neq \emptyset$ implies there exists $e = [p,q] \in \mathcal{E}(\mathcal{M})$ (with $p \neq q$) such that $e = V_v \cap V_u$; we have both $u, v \in D(p)$ and $u, v \in D(q)$. Since u and v are extreme points of both $D(p)$ and $D(q)$ and since the interiors of $D(p)$ and $D(q)$ are disjoint we have $D(p) \cap D(q) = DE(u,v)$.

Thus, if k is another Delaunay edge with $k \cap l \neq \emptyset$, then either $k \subset D(p)$ or $k \subset D(q)$ or not. In the first two cases, then $\angle(k,l) \geq \theta_0$ follows from Claim 6.10. In the third case we suppose k is not a subset of either $D(p)$ or $D(q)$. However, if $\angle(k,l) < \theta_0$ then the "interior" of k would intersect the interior of $D(p)$ or $D(q)$. See Figure 6. Thus, we must have $\angle(k,l) \geq \theta_0$. ■

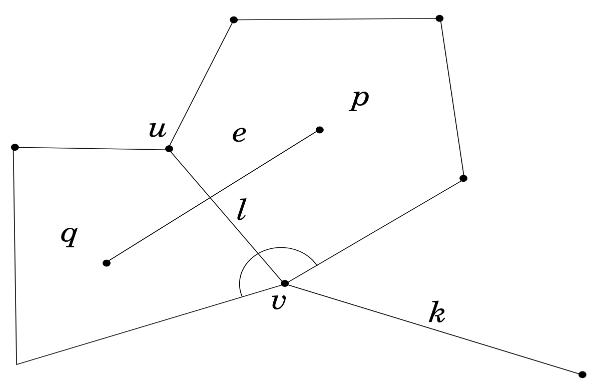
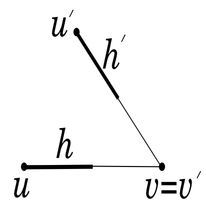


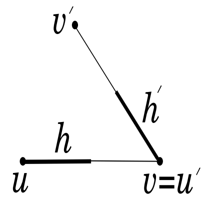
Figure 6. The Intersection of Trees

Proof of 6.8. Let m_K be from Corollary 6.4.2 and $m \geq m_K$. For each Delaunay half edge h there is a unique Delaunay edge containing it; if $h = [u, p_{uv}]$ (or $h = [v, p_{uv}]$) we write $Dh = [u,v]$. If $Dh \cap Dh' = D[u, p_{uv}] \cap D[u', p_{u'v'}] = \emptyset$, then by Corollary 6.4.2 $D[u, p_{uv}]_K \cap D[u', p_{u'v'}]_K = \emptyset$ implying $[u, p_{uv}]_K \cap [u', p_{u'v'}]_K = \emptyset$. If $D[u, p_{uv}] \cap D[u', p_{u'v'}] \neq \emptyset$ and $u \neq u'$, then there are three cases; *i)* $v = v'$, *ii)* $v = u'$, or *iii)* $v' = u$. See Figure 7. Let θ_0 be from Claim 6.11.

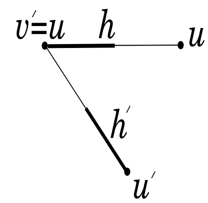
Figure 7. The Three Cases in the Proof of 6.8.



Case *i.*



Case *ii.*



Case *iii.*

Case i. Let $k = [u, v']$ and $l = [v', u']$. Then $\angle(k, l) = \angle(u, v', u') \geq \theta_0$. For $m \geq m'_K > 2K/\sin(\theta_0/2)$, for any two points $p \in k$ and $q \in l$ with $d(p, v'), d(q, v') \geq m/2$, the distance $d(p, q) \geq m \sin(\theta_0/2) > 2K$. In particular, $[u, p_{uv}]_K \cap [u', p_{u'v'}]_K = \emptyset$.

Case ii. Let $l = [u, v]$, $k = [v, v']$ and $m \geq m''_K > 4K/\sin(\theta_0)$. If $u \neq v'$, then $\angle(l, k) \geq \theta_0$ and it is easy to check that for $p \in l = [u, v]$ with $d(p, v) \geq m/2$ that $d(p, k) > 2K$ and thus that $[u, p_{uv}]_K \cap [v, p_{vv'}]_K = \emptyset$. If $u = v'$, then $l = Dh = Dh' = k$ and $h_K \cap h'_K = \overline{B}(p, K)$ where $p = h \cap h' = p_{uv} = p_{vv'}$.

Case *iii* is identical to Case *ii*, thus taking m_K suitably large, 6.8 is proven. ■

Part 5) follows by the calculation in Case *i* above, completing the proof of Theorem 6.1.

■

7. Coding Thick Linear Graphs.

The primary result in this section tells us that if Y is a square mixing SFT and m and K are large enough, then it is possible to use $\mathfrak{T}_m$ and $\mathcal{T}_m(\mathcal{M})$ to produce locally Y allowed words on the lattice points inside the K thickened linear graphs, $DG(\mathcal{M})_K \subset \mathbb{R}^2$ for $\mathcal{M} \in \mathfrak{M}_m$, in a continuous and shift commuting manner.

Because we are now discussing words which occur on infinite subsets of $\mathbb{Z}^2$, we need a topology on collections of such words. Moreover, we want two words $u \in W_Y(R)$ and $v \in W_Y(T)$ to be close together if, both, the sets R and T agree in some (big) square Λ_N and the symbols in the words u and v agree on the set $\Lambda_N \cap R = \Lambda_N \cap T$. This motivates the next definition.

If S is a finite symbol set, then $\bigcup_{\alpha \in \mathcal{P}(\mathbb{Z}^2)} S^\alpha$ is the set of all words which can appear on subsets of $\mathbb{Z}^2$. In this context, let $\mathcal{U}(\alpha, B) = \{\beta \in \mathcal{P}(\mathbb{Z}^2) : \beta \cap B = \alpha \cap B\}$ for $B \subset \mathbb{Z}^2$ finite and $\alpha \in \mathcal{P}(\mathbb{Z}^2)$.

Definition 7.1. The *Bounded Neighborhood* (BN) topology on $\bigcup_{\alpha\in\mathcal{P}(\mathbb{Z}^2)} S^\alpha$ is that topology generated by the collection of all subsets

$$\mathcal{U}(w,\alpha,B)=\{w'\in S^\beta : w'\big|_B = w\big|_B,\ \forall\,\beta\in\mathcal{U}(\alpha,B)\}$$

for every finite subset $B\subset\mathbb{Z}^2$, $\alpha\in\mathcal{P}(\mathbb{Z}^2)$ and $w\in S^\alpha$.

In the next theorem it is more natural and effective to use the language of square mixing rather than square filling. In Section 8 we show that a square filling mixing SFT is square mixing, so the theorem will apply to square filling mixing SFTs as well.

Theorem 7.2. *Given $K,k>0$, suppose*

1) Y is a $\mathbb{Z}^2$ square mixing matrix SFT with square mixing parameter $\widetilde{k}$,

2) $K'\geq K+2k\geq 4(\widetilde{k}+2)$, and

3) $m\geq m_{2K'}$ where $m_{2K'}$ is from Theorem 6.1.

Then there exists a continuous shift commuting map

$$\Theta:\mathfrak{M}_m\to\{W_Y^{loc([1,k]^2)}(DG(\mathcal{M})_K):\mathcal{M}\in\mathfrak{M}_m\}.$$

That is, for $\mathcal{M}\in\mathfrak{M}_m$ each image $\Theta(\mathcal{M})$ is a $[1,k]^2$ locally Y allowed word on $DG(\mathcal{M})_K$.

Remark 7.3. Because Y is assumed to be only square mixing and not necessarily square filling, the image $\Theta(\mathcal{M})$ is only a locally Y allowed word on the thickened linear graph $DG(\mathcal{M})_K$. This word may not occur in any point in Y - to guarantee occurrence we need to assume Y is square filling as well. It is here and only here that we cannot avoid the assumption of square filling.

To prove Theorem 7.2 it suffices to prove the following lemma.

Lemma 7.4. *Under the assumptions (1, 2, and 3) of Theorem 7.2 there exists a map*

$$\Theta:\mathfrak{T}_m\to\{W_Y(T_K):T\in\mathfrak{T}_m\}$$

such that for $\mathcal{M}\in\mathfrak{M}_m$ and for any pair $(T,v),(T',v')\in\mathcal{T}_m(\mathcal{M})$ we have $\sigma^{-v}\Theta(T)$ and $\sigma^{-v'}\Theta(T')$ agree on $(T_K+v)\cap(T'_K+v')$ and the word

$$\Theta=\Theta(T_K+v\cup T'_K+v')\equiv\begin{cases}\sigma^{-v}\Theta(T) & \text{on}\quad T_K+v\\ \sigma^{-v'}\Theta(T') & \text{on}\quad T'_K+v'\end{cases}$$

is a $[1,k]^2$ *locally* Y *allowed word on* $T_K + v \cup T'_K + v'$.

Before we prove Lemma 7.4 let us use it to prove Theorem 7.2.

Proof of 7.2. For $\mathcal{M} \in \mathfrak{M}_m$, we define the image $\Theta(\mathcal{M})$ as follows:

$$\Theta(\mathcal{M}) \equiv \bigcup_{(T,v)\in\mathcal{T}_m(\mathcal{M})} \sigma^{-v}\Theta(T). \tag{5}$$

Because Lemma 7.4 guarantees $\sigma^{-v}\Theta(T)$ and $\sigma^{-v'}\Theta(T')$ agree on $(T_K + v) \cap (T'_K + v')$ for every (T,v), $(T',v') \in \mathcal{T}_m(\mathcal{M})$, it follows that $\Theta(\mathcal{M})$ is a well defined word on $DG(\mathcal{M})_K$.

Next we will show $\Theta(\mathcal{M})$ is $[1,k]^2$ locally Y allowed. We first prove that $[1,k]^2 + t$ intersects at most two thickened tiles $(T+d)_K$ and $(T'+d')_K$, where $(T,d), (T',d') \in \mathcal{T}_m(\mathcal{M})$. Let $T + d$, $T' + d'$ and $T'' + d''$ be three distinct tiles and let $p' = T + d \cap T' + d'$ and $p'' = T + d \cap T'' + d''$. Because the intersection of any two tiles $T + d$ and $T' + d'$ is at most a point and is a midpoint of a Delaunay edge in $DG(\mathcal{M})$ (Theorem 6.1, Part 3), it follows that any point p can be in at most two tiles. Thus $p' \neq p''$.

When $m \geq m_{2K'}$, this also holds for K' thickened tiles. Suppose $q \in (T + d)_{K'} \cap (T' + d')_{K'} = \overline{B}(p', K')$. Then q cannot be an element of any other thickened tile $(T''+d'')_{K'}$ since this would imply that $q \in (T+d)_{K'} \cap (T''+d'')_{K'} = \overline{B}(p'', K')$ and thus that $d(p', p'') \leq 2K'$, contradicting the separation of the terminal vertices of $T + d$ (Theorem 6.1, Part 5).

So, if $([1,k]^2 + t)$ has a nonempty intersection with each of $(T + d)_K$, $(T' + d')_K$ and $(T'' + d'')_K$, then the previous paragraph will be contradicted (since $K' \geq K + 2k$). Since $[1,k]^2 + t$ intersects at most two K thickened tree tiles, if $([1,k]^2 + t) \subset DG(\mathcal{M})_K$ then $([1,k]^2 + t) \subset (T_K + v \cup T'_K + v')$ for some (T,v) and (T',v'), and because Θ is a $[1,k]^2$ locally Y allowed word on $T_K + v \cup T'_K + v'$, it follows that $\Theta(\mathcal{M})$ is $[1,k]^2$ locally Y allowed on $DG(\mathcal{M})_K$.

The continuity of the map $\mathcal{M} \mapsto \Theta(\mathcal{M})$ is clear: There exists $\eta > 0$ such that the tilings $\mathcal{T}_m(\mathcal{M})$ and $\mathcal{T}_m(\mathcal{M}')$ agree on Λ_N if $\mathcal{M}$ and $\mathcal{M}'$ agree on $\Lambda_{N+\eta}$. Since the tilings agree on Λ_N, so do the the words $\Theta(\mathcal{M})$ and $\Theta(\mathcal{M}')$.

To see Θ commutes with the shift σ, observe first by Theorem 6.1, Part 4 that

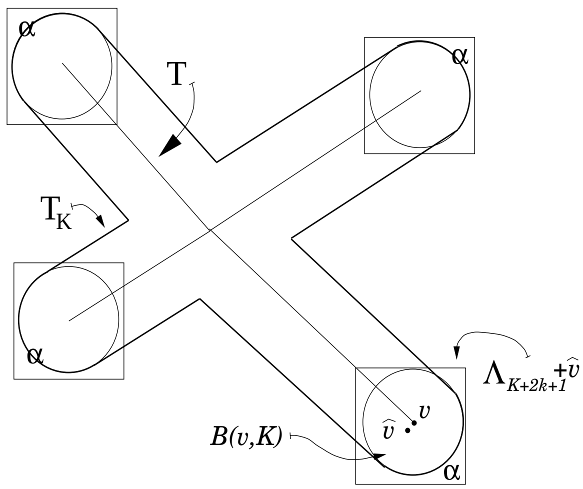
$$\mathcal{T}_m(\sigma^v\mathcal{M}) = \sigma^v\mathcal{T}_m(\mathcal{M}) = \{(T, u - v) : (T,u) \in \mathcal{T}_m(\mathcal{M})\}.$$

So,

$$\Theta(\sigma^v \mathcal{M}) = \bigcup_{(T',u')\in\mathcal{T}_m(\sigma^v\mathcal{M})} \sigma^{-u'}\Theta(T') = \bigcup_{(T,u)\in\mathcal{T}_m(\mathcal{M})} \sigma^{-u+v}\Theta(T) = \sigma^v\Theta(\mathcal{M})$$

which completes the proof of Theorem 7.2. ∎

FIGURE 8. A Word on a Prototile.



Proof of 7.4: We first pick and fix a word $\alpha \in W_Y(\Lambda_{K+2k+1})$. For a point $v \in \mathbb{R}^2$ we let $\widehat{v} \in \mathbb{Z}^2$ denote the unique point in $\mathbb{Z}^2$ such that $v = \widehat{v} + a$, for $a \in [0,1)^2$. For the prototile $T \in \mathfrak{T}_m$ there exists a point $y \in Y$ such that for each terminal vertex v in T we have $y|_{\Lambda_{K+2k+1}+\widehat{v}} = \alpha$. To see this observe, because $m \geq m_{2K'}$ we have (by Theorem 6.1, Part 5) $B(v, 2K') \cap B(v', 2K') = \emptyset$ for any two distinct terminal vertices, v and v' in the prototile T. Thus $B(\widehat{v}, 2K'-1) \cap B(\widehat{v'}, 2K'-1) = \emptyset$. Our choice of $K' \geq K + 2k \geq 4(\widetilde{k}+2)$ implies

$$\Lambda_{K+2k+1+\widetilde{k}} \subset \Lambda_{K'+1+\widetilde{k}} \subset \Lambda_{\frac{5}{4}K'-1} \subset B(0, 2K'-1)$$

and this guarantees (by square mixing) for each terminal vertex v and any point $x \in Y$, the existence of y such that $y|_{\Lambda_{K+2k+1}+\widehat{v}} = \alpha$ and such that y and x agree outside the square $\widehat{v} + \Lambda_{\frac{5}{4}K'-1}$ and hence y and x will agree outside the ball $B(\widehat{v}, 2K'-1)$. See Figure 8.

So we can find a point y which has a copy of the word α on each square $\Lambda_{K+2k+1}+\widehat{v}$ for each terminal vertex $v \in T$. We define

$$\Theta(T) = y|_{T_K}.$$

Thus $\Theta(T) \in W_Y(T_K)$.

Now we check the compatibility relations. Suppose for distinct $(T,d),(T',d') \in \mathcal{T}_m(\mathcal{M})$ we have $(T_K+d)\cap(T'_K+d') \neq \emptyset$, then it is easy to check the corresponding words agree on the intersection, *i.e.* $(\Theta(T)+d)|_{T'_K+d'} = (\Theta(T')+d')|_{T_K+d}$. If $(T_K+d)\cap(T'_K+d') \neq \emptyset$ and $(T,d),(T',d') \in \mathcal{T}_m(\mathcal{M})$ are distinct, then the intersection of the K thickened tiles is a ball of radius K, $\overline{B}(w,K)$, centered on the common terminal vertex, w. So, we must check that the words $\Theta(T)+d$ and $\Theta(T')+d'$ agree on $\overline{B}(w,K)$. If v and v' are terminal vertices of T and T', respectively and $w = v+d = v'+d'$ is the common terminal vertex of $T+d$ and $T'+d'$, then $\overline{B}(v+d,K) = \overline{B}(v'+d',K)$ is the K thickened tile intersection $(T_K+d)\cap(T'_K+d')$ and $v-\widehat{v} = v'-\widehat{v'}$. We check

$$\begin{aligned}(\Theta(T)+d)|_{\overline{B}(v+d,K)} = \Theta(T)|_{\overline{B}(v,K)} &= \alpha|_{\overline{B}(v-\widehat{v},K)} \\ &= \alpha|_{\overline{B}(v'-\widehat{v'},K)} = \Theta(T')|_{\overline{B}(v',K)} = (\Theta(T')+d')|_{\overline{B}(v'+d',K)}\end{aligned}$$

and we see that the words agree.

Whether each word $\Theta = \Theta(T_K+v \cup T'_K+v')$ is $[1,k]^2$ locally Y allowed requires us to check that whenever $\mathcal{S} = ([1,k]^2+u) \subset (T_K+v\cup T'_K+v')$, the word $\Theta|_{\mathcal{S}} \in W_Y(\mathcal{S})$. If $\mathcal{S} = ([1,k]^2+u) \subset (T_K+v)$ then indeed $\Theta|_{\mathcal{S}} = \sigma^{-v}\Theta(T)|_{[1,k]^2+u} = \sigma^{-v}y|_{T_K+v}|_{[1,k]^2+u} = \sigma^{-v}y|_{\mathcal{S}} \in W_Y(\mathcal{S})$, where y is from the construction of $\Theta(T)$. Suppose $([1,k]^2+u) \not\subset (T_K+v)$ and $([1,k]^2+u) \not\subset (T'_K+v')$. Then $[1,k]^2+u$ intersects both $T_K+u\backslash T'_K+v'$ and $T'_K+v'\backslash T_K+u$. This can only happen if $[1,k]^2+u\cap\overline{B}(w,K) \neq \emptyset$ for w, the terminal vertex of both $T+v$ and $T'+v'$. Thus $[1,k]^2+u \subset \overline{B}(w,K+2k)$. However, in that case $[1,k]^2+u \subset \Lambda_{K+2k+1}+\hat{w}$ and thus $\Theta|_{[1,k]^2+u}$ is a subword of α and thus $\Theta|_{\mathcal{S}} \in W_Y(\mathcal{S})$. Hence Θ is $[1,k]^2$ locally Y allowed. This completes the proof of Lemma 7.4. ∎

8. STITCHING.

In this section our principal goal will be to prove Theorem 8.1. The section will end with the proof of Lemma 2.11. Roughly speaking, the content of Theorem 8.1 is, if Y is a mixing SFT and if Y is square filling, then Y is "filling" on more general shapes, in particular, uniform families of convex polygons. The proof will proceed by means of the technique of "stitching" which gives square filling mixing SFTs their local constructibility.

The idea behind stitching is the following. Let $L = \{(v_1, v_2) \in \mathbb{Z}^2 : v_2 < 0\}$ and $U = \{(v_1, v_2) \in \mathbb{Z}^2 : v_2 \geq 0\}$. Suppose Y is a mixing square filling SFT and suppose $\alpha \in W_Y(L)$ and $\beta \in W_Y(U)$. Letting $x \in S_Y^{\mathbb{Z}^2}$ be such that $x_L = \alpha$ and $x_U = \beta$, x has a horizontal stripe of symbols which are not locally Y allowed. We use the mixing to establish that there are vertical bars of locally Y allowed symbols. See Figure 9. The square filling is used to fill in the remaining square patches with locally Y allowed words, creating an element $y \in Y$ which agrees with x away from the horizontal axis. In what follows we show this can be done for words on more general shapes.

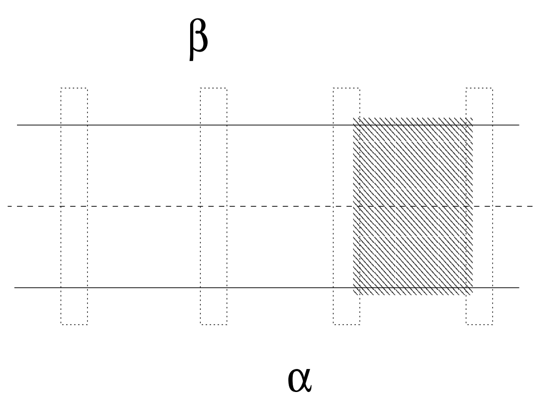


FIGURE 9. Stitching a Horizontal Line.

Recall, by a *convex polygon* in $\mathbb{R}^2$ we mean the closed convex hull P of a finite number of points in $\mathbb{R}^2$ such that the interior of P is nonempty. Let $\kappa > 1$. By a κ *uniform family* of convex polygons we mean $\mathcal{V} = \bigcup_{n \geq n_0} \mathcal{V}(n)$ where each $\mathcal{V}(n)$ is a family of convex polygons such that $V \in \mathcal{V}(n)$ implies $B(p, n/\kappa) \subset V \subset B(q, \kappa n)$ for some $p, q \in \mathbb{R}^2$.

Observe that for any square filling mixing matrix SFT Y, there exist k and l such that Y is (k, l) square filling, and because Y is mixing we may choose l large enough that for any $\hat{l} \geq l$ and for any two words $\alpha, \beta \in W_Y([0, k-1]^2)$ there exist points $y, y' \in Y$ such that y exhibits α and β separated by $(0, 2\hat{l} - k + 1)$ and y' exhibits α and β separated by $(2\hat{l} - k + 1, 0)$. We refer to such (k, l) as *generalized filling* parameters for Y. For technical reasons we will always assume $l > 3k$.

Theorem 8.1. *Let Y be a square filling, mixing matrix SFT with generalized filling parameters (k, l), let $K \geq 20l$, and let $\kappa > 1$. For any κ uniform family of convex polygons $\mathcal{V} = \bigcup_{n \geq n_0} \mathcal{V}(n)$, any $n > \kappa \cdot K$ and any $V \in \mathcal{V}(n)$, if $w \in W_Y^{loc([1,k]^2)}((\partial V)_K)$, then $w|_{(\partial V)_2} \in W_Y((\partial V)_2)$.*

We define a set γ to be an l *block boundary* if there exists a set $\mathcal{A} \subset \mathbb{Z}^2$ such that $\gamma = \partial D + w$ where $D = \bigcup_{v \in \mathcal{A}} \Lambda_{5l} + 10lv$ and $w \in \mathbb{Z}^2$. (Recall $\Lambda_{5l} = \{v \in \mathbb{R}^2 : \|v\|_{\sup} \leq 5l\}$.) Consequently, an l block boundary is the union of a set of orthogonal line segments $10l$ in length which only intersect at their endpoints. If γ is an l block boundary, then we will refer to $c \in \gamma$ as a *corner (in γ)* if two of the line segments intersect at c and are perpendicular to each other. Thus, if γ is an l block boundary we can write $\gamma = \cup L_i$ where each L_i is a line segment which is a multiple of $10l$ in length, and $\cup \mathcal{V}(L_i)$ is the set of corners in γ (where $\mathcal{V}(L_i)$ denotes the endpoints (*v*ertices) of L_i). If, in addition, γ is homeomorphic to a circle, then we may assume $L_i \cap L_j \neq \emptyset \iff j = i \pm 1$ and $L_i \cap L_{i+1} = c_{i+1}$ (where each $L_i = [c_i, c_{i+1}]$).

For a convex polygon V and $K > 0$ define $V_K^- = \{v \in V : d(v, \partial V) \geq K\}$. V_K^- is a closed convex set and if the interior of V_K^- is nonempty, then V_K^- is also a convex polygon.

Claim 8.2. *Given an integer $l > 0$ if $K \geq 20l$, then for any convex polygon V for which $V_K^- \neq \emptyset$, there exists a set D for which ∂D is an l block boundary and such that $V_K^- \subset D \subset V$. In fact, we may take D to be a subset of the interior of V_{4l}^- and such that ∂D is homeomorphic to a circle.*

Proof of 8.2. For some finite $\mathcal{A} \subset \mathbb{Z}^2$, $V_K^- \subset \bigcup_{v \in \mathcal{A}} (\Lambda_{5l} + 10lv)$. If for some $v \in \mathcal{A}$ the set $\Lambda_{5l} + 10lv$ is not contained in the interior of V_{4l}^-, then we may replace $\mathcal{A}$ with $\mathcal{A}' = \mathcal{A} \backslash \{v\}$ because for such a v the square $\Lambda_{5l} + 10lv$ does not contribute to the cover of V_K^- (since $\Lambda_{5l} + 10lv \cap V_K^- = \emptyset$). A finite number of such replacements will result in a set

$D = \bigcup_{v\in\mathcal{A}} \Lambda_{5l} + 10lv$ which has the properties that $D \subset V_{4l}^-$ and the boundary of D is an l block boundary. The reader can check that because V is convex, for D such that ∂D is not homeomorphic to the circle one may add blocks (the sets $\Lambda_{5l} + 10lv$) to form D' for which $\partial D'$ is homeomorphic to the circle and yet retain $V_K^- \subset D' \subset Interior(V_{4l}^-)$. ■

Proof of 8.1. Suppose $n > \kappa \cdot K$ and $V \in \mathcal{V}(n)$; hence $V_K^- \neq \emptyset$. Let $D \subset Interior(V_{4l}^-)$ be as from Lemma 8.2 (with ∂D homeomorphic to a circle). So $\gamma = \partial D = \bigcup L_i$ where $L_i \cap L_j \neq \emptyset \iff j = i \pm 1$, each $L_i = [c_i, c_{i+1}]$ is an integer multiple of $10l$ in length and the set of c_i's is the set of corners in γ. If $w \in W_Y^{loc([1,k]^2)}((\partial V)_K)$ then the restriction $u = w|_{(\partial V)_K\backslash D}$ is an element of $W_Y^{loc([1,k]^2)}((\partial V)_K\backslash D)$. Let $v = x|_D$ for any $x \in Y$. Let U denote the word on V_K which equals u and v on the appropriate restrictions. We will now proceed to construct a word in $W_Y^{loc}(V_K)$ which agrees with U (and hence w) on $(\partial V)_2$.

Elementary in this is the stitch. There are three types of stitches: corner stitches, horizontal stitches, and vertical stitches. The purpose of a stitch is to alter the symbols in a square $\Lambda_l + v$ to form a word which is locally Y allowed on Λ_l. We will begin by demonstrating how a corner stitch is made. Then we will demonstrate how the application of horizontal and vertical stitches to $v \in \gamma \cap \mathbb{Z}^2$ for v between corners creates locally Y allowed words between the corners. The repetition of these techniques will produce a locally Y allowed word in a neighborhood of γ without altering the word outside the neighborhood, thus leaving us with a locally Y allowed word on V_K which agrees with U on V_2.

Suppose $c \in \gamma \cap \mathbb{Z}^2$ is a corner and suppose the two line segments L_i and L_{i+1} which intersect at c are $L_i = [c, c + (0, -m)]$ and $L_{i+1} = [c, c + (m', 0)]$ for some $m, m' > 0$. That is, one line segment extends down and the other extends to the right from c. For ease of notation, suppose that $c = \overline{0}$, the origin in $\mathbb{Z}^2$. Because γ is an l block boundary and homeomorphic to a circle $\Lambda_l \cap \gamma = [\overline{0}, (0, -l)] \cup [\overline{0}, (l, 0)]$ and $\Lambda_l \backslash \gamma = \Lambda_l \backslash ([\overline{0}, (0, -l)] \cup [\overline{0}, (l, 0)])$. The word $U|_{\Lambda_l}$ need not be locally Y allowed, indeed this is what we wish to remedy, however $U|_{\Lambda_l\backslash\gamma}$ *is* $[1, k]^2$ locally Y allowed.

Claim 8.3. *(A Corner Stitch.) There exists a locally Y allowed word on Λ_l which agrees with U on $\Lambda_l \backslash (\Lambda_{l-1} \cup \Gamma_{l-k})$ where $\Gamma_{l-k} = \cup\{\Lambda_{l-k} + v : v \in \gamma \cap \mathbb{Z}^2\}$.*

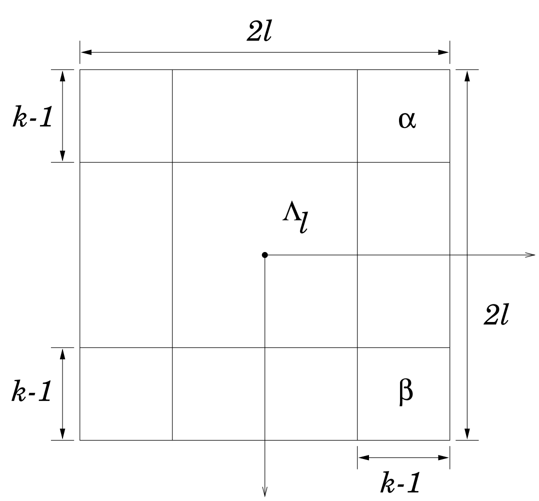
There are similar corner stitches for the other three ways a pair of line segments can intersect at a corner in γ.

Proof of 8.3. Briefly, we must first use the fact that Y is mixing to produce a locally Y allowed word on $\Lambda_l \backslash \Lambda_{l-k}$, then square filling will allow us to fill this word in, completing the proof of the claim.

If we examine the $2l \times k-1$ rectangle R_T sitting across the top of the square Λ_l (see Figure 10) we see that R_T is a subset of V_K which sits outside of Γ_{l-k} because γ is an l block curve. This means that U restricted to R_T is $[0,k-1]^2$ locally Y allowed, because the rectangle is a subset of the region in V_K where U is $[0,k-1]^2$ locally Y allowed. In a similar fashion, U on the $k-1 \times 2l$ rectangle R_L sitting on the left of the square Λ_l is also $[0,k-1]^2$ locally Y allowed.

Now we examine the $k-1 \times 2l$ rectangle R_R sitting on the right hand side of the square Λ_l. If $U|_{R_R}$ is locally Y allowed, then we do nothing. If, on the other hand, $U|_{R_R}$ is not locally Y allowed then we examine the two $k-1 \times k-1$ square words $\alpha = U|_{R_R \cap R_T}$ and $\beta = U|_{R_R \cap R_B}$ where R_B is the $2l \times k-1$ rectangle sitting across the bottom of the square Λ_l. See Figure 10. Now U restricted to each of these $[0,k-1]^2$ squares is a Y allowed word *which occurs*

FIGURE 10. Constructing a Corner Stitch.



because each $[0,k-1]^2$ square does not intersect Γ_{l-k}. We have chosen l to be large enough so that there exists $y \in Y$ with $y|_{R_R \cap R_B} = \beta$ and $y|_{R_R \cap R_B + (0,2l-k+1)} = y|_{R_R \cap R_T} = \alpha$. Thus $y|_{R_R}$ is a locally Y allowed word such that $y|_{R_R \cap R_T} = U|_{R_R \cap R_T}$ and $y|_{R_R \cap R_B} = U|_{R_R \cap R_B}$.

($N.B.$, In order for y to exist, α and β must occur; it is for this reason that we begin with $w \in W_Y^{loc([1,k]^2)}((\partial V)_K)$.)

By a similar means we obtain a point $y' \in Y$ such that $y'|_{R_B \cap R_R} = U|_{R_B \cap R_R}$ and $y'|_{R_B \cap R_L} = U|_{R_B \cap R_L}$. We define the word W on $\Lambda_l \backslash \Lambda_{l-k}$ as follows

$$W = \begin{cases} U|_{R_L \cup R_T} & \text{on } R_L \cup R_T \\ y|_{R_R} & \text{on } R_R \\ y'|_{R_B} & \text{on } R_B. \end{cases}$$

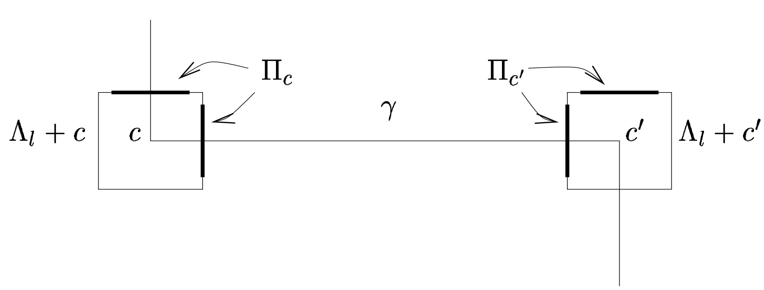
Our choice of y and y' guarantees that $W \in W_Y^{loc}(\Lambda_l \backslash \Lambda_{l-k})$ and that $W|_{\Lambda_l \backslash (\Lambda_{l-k} \cup \Gamma_{l-k})} = U|_{\Lambda_l \backslash (\Lambda_{l-k} \cup \Gamma_{l-k})}$. Square filling ensures the existence of $W' \in W_Y^{loc}(\Lambda_l)$ such that $W'|_{\Lambda_l \backslash \Lambda_{l-1}} = W|_{\Lambda_l \backslash \Lambda_{l-1}}$. Hence $W'|_{\Lambda_l \backslash (\Lambda_{l-1} \cup \Gamma_{l-k})} = U|_{\Lambda_l \backslash (\Lambda_{l-1} \cup \Gamma_{l-k})}$. ■

Let $\mathcal{C} \subset \gamma$ be the collection of corners in γ. By Claim 8.3, for each $c \in \mathcal{C}$ there exists a word $w_c \in W_Y^{loc}(\Lambda_l + c)$ such that w_c agrees with U on $(\Lambda_l \backslash \Lambda_{l-1} + c) \backslash \Gamma_{l-k}$. We define a new word U' on V_K as follows

$$U' = \begin{cases} w_c & \text{on } \Lambda_l + c \text{ for } c \in \mathcal{C} \\ U & \text{on } V_K \backslash \bigcup_{c \in \mathcal{C}} \Lambda_l + c \end{cases}.$$

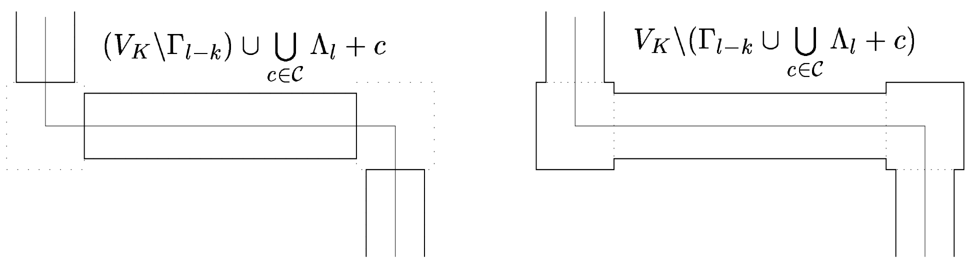
Because $\gamma = \partial D$ and $D \subset V_{4l}^-$, it follows that $U' = U$ on $V_K \backslash Interior(V_{2l}^-)$. Because each w_c is locally Y allowed on $\Lambda_l + c$, the word U' is locally allowed in places that U was not, however, since w_c does not necessarily agree with U on all of $\Lambda_l \backslash \Lambda_{l-1} + c$, U' need not be locally Y allowed in places that U was. We should elaborate on this. See Figure 11.

FIGURE 11. Where U' is not locally allowed.



Let $P_v^1 \equiv [v, v+e_1]$ and $P_v^2 \equiv [v, v+e_2]$ where $e_i : i = 1, 2$ are the usual basis vectors generating $\mathbb{Z}^2$. According to our construction of U, for every $v \in \mathbb{Z}^2$ for which $P_v^i \subset V_K \backslash D$ or $P_v^i \subset D$, the word $U|_{P_v^i} \in W_Y(P_v^i)$ for $i = 1, 2$. With the word U' we have gained a little and lost a little. As a gain we now have, if $P_v^i \subset \Lambda_l + c$ for any $c \in \mathcal{C}$, then $U'|_{P_v^i} \in W_Y(P_v^i)$ for $i = 1, 2$. To explain the loss let $\Pi_c = (\Lambda_l \backslash \Lambda_{l-1} + c) \cap \Gamma_{l-k}$ for $c \in \mathcal{C}$. If P_v^i intersects Π_c, but is not a subset of $\Lambda_l + c$, then $U'|_{P_v^i}$ need not be an element of $W_Y(P_v^i)$. (If P_v^i intersects $\gamma \backslash \cup_{c \in \mathcal{C}} \Lambda_l + c$ but is not a subset of D, then $U'|_{P_v^i}$ need not be an element of $W_Y(P_v^i)$ either, however that was true for U as well). We could document precisely for which v the word $U'|_{P_v^i}$ need not occur in Y, but this is unnecessary. It will suffice to observe the following (see Figure 12).

FIGURE 12. Where U' is locally allowed.



Remark 8.4. *The word U' is locally Y allowed on $(V_K \backslash \Gamma_{l-k}) \cup \bigcup_{c \in \mathcal{C}} \Lambda_l + c$ and $[1, k]^2$ locally Y allowed on $V_K \backslash (\Gamma_{l-k} \cup \bigcup_{c \in \mathcal{C}} \Lambda_l + c)$.*

The set $\gamma \backslash (\bigcup_{c \in \mathcal{C}} \Lambda_l + c)$ is a disjoint union of horizontal and vertical line segments. We now use horizontal stitches to alter U' near each horizontal line segment and vertical stitches to alter U' near each vertical line segment producing the desired locally Y allowed word on V_K. We demonstrate this on a horizontal line segment $L \subset \gamma$. The construction of stitches on vertical line segments works analogously.

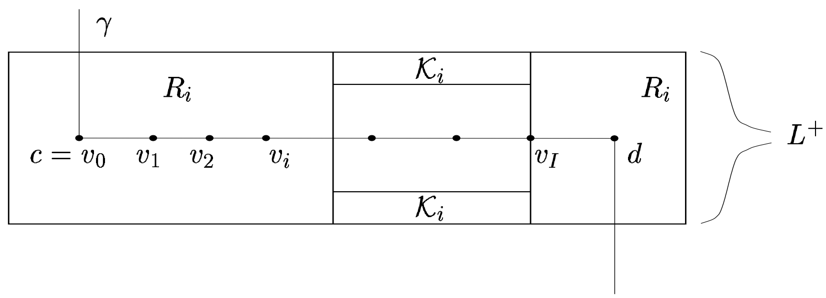
Recall, $\gamma = \cup L_i$ where the L_i are line segments which are a multiple of $10l$ in length and either horizontal or vertical. Let $L = L_i = [c, d]$ be such a horizontal line segment, where c and d are corners in γ with c sitting to the left of d. We may select a set $H \subset [c + (l, 0), d - (l, 0)] \cap \mathbb{Z}^2$ which has the following properties. 1) If $h \in H$, then no other element of H is within a distance k of h. 2) Every element of $[c + (l, 0), d - (l, 0)] \cap \mathbb{Z}^2$ is within a distance k of some element of H. We let $H = \{v_i\}_{i=1}^{I-1}$ be an ordering of H from left to right (so v_i is to the left of v_{i+1}) and let $v_0 = c$ and $v_I = d - (l, 0)$. See Figure 13.

We now define three regions. For $i = 0, .., I$ write

$$R_i = (\Lambda_l + d) \cup \bigcup_{j=0}^{i} (\Lambda_l + v_j)$$
$$L^+ = \bigcup_{v \in L} (\Lambda_l + v)$$
$$\mathcal{K}_i = L^+ \backslash (R_i \cup \Gamma_{l-k}).$$

So $R_i \subset L^+$ and $R_I = L^+$.

FIGURE 13. Constructing Horizontal Stitches.



By Remark 8.4, we know U' is locally Y allowed on $R_0 \cup \mathcal{K}_0$ and is $[1, k]^2$ locally Y allowed on $\mathcal{K}_0$.

Claim 8.5. *(A Horizontal Stitch) For $i \in \{0, .., I-1\}$, suppose U' is a word locally Y allowed on $R_i \cup \mathcal{K}_i$ and $[1, k]^2$ locally Y allowed on $\mathcal{K}_i$.*

1) If $R_{i+1} \neq L^+$, then there exists a word U'' which is locally Y allowed on $R_{i+1} \cup \mathcal{K}_{i+1}$ and $[1, k]^2$ locally Y allowed on $\mathcal{K}_{i+1}$ and agrees with U' on $L^+ \backslash \left(\bigcup_{v \in L} \Lambda_{l-1} + v\right)$.

2) If $R_{i+1} = L^+$, then there exists a word U'' which is locally Y allowed on L^+ and agrees with U' on $L^+ \backslash \left(\bigcup_{v \in L} \Lambda_{l-1} + v\right)$.

Proof of 8.5. The proof of this claim follows much as the proof of Claim 8.3. We will follow the notation introduced in the proof of Claim 8.3 for R_L, R_R, R_T, and R_B as the rectangles on the left, right, top and bottom of Λ_l, respectively.

Part 1). For notational ease let us suppose $v_{i+1} = \overline{0}$. So $R_{i+1} = R_i \cup (\Lambda_l + v_{i+1}) = R_i \cup \Lambda_l$. Our choice of H (specifically Part 2, which can be expressed as $\|v_{i+1} - v_i\| \leq 2k+1$) implies that $R_L, R_T, R_B \subset R_i \cup \mathcal{K}_i$ and thus that $U'|_{R_L \cup R_T \cup R_B} \in W_Y^{loc}(R_L \cup R_T \cup R_B)$. Since

$R_{i+1} \neq L^+$ and $k < \|v_{i+1} - v_i\|$ (by Part 1 in the choice of H) this implies $R_R \cap R_T$, $R_R \cap R_B \subset \mathcal{K}_i$ and it follows that $U'|_{R_R \cap R_T}$ and $U'|_{R_R \cap R_B}$ are $[1,k]^2$ locally Y allowed. By mixing there exists $y \in Y$ such that $y|_{R_R \cap R_T} = U'|_{R_R \cap R_T}$ and $y|_{R_R \cap R_B} = U'|_{R_R \cap R_B}$. Thus the word

$$W = \begin{cases} U' & \text{on } R_L \cup R_T \cup R_B \\ y & \text{on } R_R \end{cases}$$

is an element of $W_Y^{loc}(\Lambda_l \backslash \Lambda_{l-k})$ and by square filling there exists $W' \in W_Y^{loc}(\Lambda_l)$ such that W' agrees with W on $\Lambda_l \backslash \Lambda_{l-1}$. We define

$$U'' = \begin{cases} U' \text{ on } L^+ \backslash \Lambda_l \\ W' \text{ on } \Lambda_l \end{cases}.$$

Since W agrees with W' on $\Lambda_l \backslash \Lambda_{l-1}$ and since W agrees with U' on $R_R \cup R_T \cup R_B$, it follows that U'' is locally Y allowed on $R_{i+1} \cup \mathcal{K}_{i+1}$ and that U'' agrees with U' on $L^+ \backslash \cup_{v \in L} \Lambda_{l-1} + v$. Since U'' agrees with U' on $\mathcal{K}_{i+1}$, it follows that U'' is $[1,k]^2$ locally Y allowed on $\mathcal{K}_{i+1}$, completing Part 1.

Part 2). Let $i \in \{1,..,I\}$ be such that $R_{i+1} = L^+$ but $R_i \neq L^+$. Then, (since $l > 3k$) $i+2 \leq I$ and $R_{i+2} = L^+$ also. For convenience we assume $v_{i+2} = \overline{0}$, so $L^+ = R_i \cup \Lambda_l = R_{i+2}$. Moreover, because $l > 3k$, $R_R \subset \Lambda_l + d$ where R_R denotes the right hand rectangular subsets of $\Lambda_l + v_{i+2} = \Lambda_l$. As in Part 1, R_L, R_T, $R_B \subset R_i$. Thus we have $U'_{(\Lambda_l \backslash \Lambda_{l-k})}$ is locally Y allowed and by square filling there is a word $W' \in W_Y^{loc}(\Lambda_l)$ which agrees with U' on $(\Lambda_l \backslash \Lambda_{l-1})$. So,

$$U'' = \begin{cases} W' & \text{on } \Lambda_l \\ U' & \text{on } L^+ \backslash \Lambda_l \end{cases}$$

is locally Y allowed and agrees with U' on $L^+ \backslash \Lambda_{l-1}$ and thus U'' is locally Y allowed on L^+ and agrees with U' on $L^+ \backslash \bigcup_{v \in L} \Lambda_{l-1} + v$, completing Part 2. ■

By Claim 8.5, for each line segment L_i which is horizontal there is a word U_i on L_i^+ which agrees with U' on $L_i^+ \backslash \bigcup_{v \in L_i} \Lambda_{l-1} + v$, thus we may replace U' on L_i^+ with U_i. A similar result may be applied to vertical line segments. After repeated applications this results in a locally Y allowed word U'' on V_K which agrees with U on $V_K \backslash \Gamma_{l-1}$. Thus, $U''|_{\partial V_2} = w|_{\partial V_2}$.

Now, U'' is only locally Y allowed on V_K. A similar construction may be used to "stitch" U'' to a word $x''|_{\mathbb{Z}^2 \backslash D'}$ where $D' \supset V_{4l}$ has an l block boundary. This would result in a point

$y \in Y$ such that $y|_{\partial V_2} = w|_{\partial V_2}$. Thus we may conclude that $w|_{\partial V_2} \in W_Y(\partial V_2)$ completing the proof of Theorem of 8.1. ■

We finish this section with the proof of Lemma 2.11. In Section 11 we present an example demonstrating that the converse of the lemma is false. With the "stitching" technique now available the proof of Lemma 2.11 is straightforward.

Lemma 2.11: *Let Y be a mixing SFT. If Y is square filling, then Y is square mixing.*

Proof: Suppose Y is a square filling mixing matrix SFT with generalized filling parameters (k, l). Let $\widetilde{k} = 2l + 1$, $n \geq 0$, and let $\gamma = \partial\Lambda_{n+l+1}$. For every lattice point $v \in \gamma$ we have $\Lambda_l + v \subset \Lambda_{\widetilde{k}+n} \backslash \Lambda_n$. Choosing $x, y \in Y$ then we can restrict y to the interior and x to the exterior of Λ_{n+l+1} forming $[0, k-1]^2$ locally Y allowed words. Then we apply the stitching technique to this curve γ (which sits as a "fracture" in between the two words $x|_{\mathbb{Z}^2 \backslash \Lambda_{n+l+1}}$ and $y|_{\Lambda_{n+l}}$.) We apply four corner stitches and then apply horizontal stitches to the two horizontal line segments in γ and apply vertical stitches to the (remaining) two vertical line segments. As we stitch, the regions Λ_n and $\mathbb{Z}^2 \backslash \Lambda_{n+\widetilde{k}}$ are not altered and when we are done stitching our way around the square γ, we have a locally Y allowed point z agreeing with x on $\mathbb{Z}^2 \backslash \Lambda_{n+\widetilde{k}}$ and agreeing with y on Λ_n. Thus Y is square mixing with parameter $\widetilde{k}$. ■

9. The Proof of Theorem 2.8: Constructing a Homomorphism.

We now combine the results of Lemmas 3.2, 5.1, and Theorems 6.1, 7.2, and 8.1 in order to construct a homomorphism $\phi : X \to Y$ and complete the proof of Theorem 2.8. There are a series of parameters whose values must be arranged in order for these results to hold simultaneously.

We begin by letting κ be as given in Lemma 5.1. We assume Y is a square filling mixing matrix SFT with generalized filling parameters (k, l) (with $l > 3k$) and square mixing parameter $\widetilde{k}$. We let $K \geq 20l$, $K' \geq K + 2k \geq 4(\widetilde{k} + 2)$ and $m \geq \max\{m_{2K'}, \kappa \cdot K\}$, where $m_{2K'}$ is from Theorem 6.1. Finally, Lemma 3.2 gives a value for M such that for any $F = F(m, M)$ constructed with Algorithm 3.1 and $x \in X$, the set $\mathcal{M}_x = \{v \in \mathbb{Z}^2 : \sigma^v x \in F\} \in \mathfrak{M}_m$. As noted (with F fixed) the map $x \mapsto \mathcal{M}_x$ is continuous and shift commuting.

According to Lemma 5.1, the set $\mathfrak{D}_m$ is a finite set of convex polygonal prototiles which has the property that for $D \in \mathfrak{D}_m$, $B(p, m/\kappa) \subset D \subset B(q, \kappa m)$ for some $p, q \in \mathbb{R}^2$, and there exists a continuous shift commuting map $\mathcal{M} \mapsto \mathcal{D}_m(\mathcal{M})$ from $\mathfrak{M}_m$ to the set of regular $\mathfrak{D}_m$ coverings. According to Theorems 7.2 and 6.1, since $K' \geq K + 2k \geq 4(\widetilde{k} + 2)$ and $m \geq m_{2K'}$, there is a continuous shift commuting map Θ and for $\mathcal{M} \in \mathfrak{M}_m$

$$\Theta : \mathcal{M} \mapsto \Theta(\mathcal{M}) \in W_Y^{loc([1,k]^2)}(DG(\mathcal{M})_K).$$

Since $x \mapsto \mathcal{M}_x$ is a map into $\mathfrak{M}_m$ which also continuous and shift commuting, the composition

$$x \mapsto \Theta(\mathcal{M}_x) \in W_Y^{loc([1,k]^2)}(DG(\mathcal{M}_x)_K)$$

is continuous and shift commuting and the image $\Theta(\mathcal{M}_x)$ is a $[1, k]^2$ locally Y allowed word on $DG(\mathcal{M}_x)_K$.

Since $\partial D_K + v = (\partial D + v)_K$ and $(DG(\mathcal{M}))_K = \bigcup_{(D,v) \in \mathcal{D}_m(\mathcal{M})} (\partial D + v)_K$, we can define the following two words for each $(D, v) \in \mathcal{D}_m(\mathcal{M})$.

$$\partial\Theta_{\mathcal{M}}(D, v)_K \equiv \Theta(\mathcal{M})\big|_{\partial D_K + v} \in W_Y^{loc([1,k]^2)}(\partial D_K + v) \tag{6}$$

and

$$\partial\Theta_{\mathcal{M}}(D, v) \equiv \Theta(\mathcal{M})\big|_{\partial D_2 + v} \in W_Y^{loc}(\partial D_2 + v), \tag{7}$$

(the latter simply being a restriction of the former).

Notice the word $\partial\Theta_{\mathcal{M}}(D, v)$ is only locally allowed in Y and isn't necessarily exhibited. To be of use it needs to be exhibited, that is we need $\partial\Theta_{\mathcal{M}}(D, v) \in \mathcal{W}_Y(\partial D_2)$. Fortunately, we have Theorem 8.1. Because of our choice of parameters ($K \geq 20l$ and $m \geq \kappa \cdot K$) and because the $\{\mathfrak{D}_m\}$ form a κ uniform family of convex polygons, the assumptions of Theorem 8.1 hold. Thus Equation 6 and Theorem 8.1 allow us to conclude

$$\partial\Theta_{\mathcal{M}}(D, v) \equiv \Theta(\mathcal{M})\big|_{\partial D_2 + v} \in W_Y(\partial D_2 + v), \tag{8}$$

a strengthening of Equation 7.

We define a map $\Phi(D, \cdot)$ which takes $w \in W_Y(\partial D_2)$ (words which occur) to $\Phi(D, w) \in W_Y(D_2)$ such that $\Phi(D, w)|_{\partial D_2} = w$. Namely, for $w \in W_Y(\partial D_2)$ there exists $x_w \in Y$ such that $w = x_w\big|_{\partial D_2}$. We define $\Phi(D, w) = x_w\big|_{D_2}$. If $w \in W_Y(\partial D_2 + v)$ we will write

$\Phi(D+v,w)$ to clarify that we intend the image to be an element of $W_Y(D_2+v)$. That is, if $w \in W_Y(\partial D_2+v)$, then $\Phi(D+v,w) = \sigma^{-v}\Phi(D,\sigma^v w)$. Letting $D' = D+v$, then we can also write $\sigma^{-v}\Phi(\sigma^v D', \sigma^v w) = \Phi(D', w)$.

So, for each $x \in X$ and for every $(D,v) \in \mathcal{D}_m(\mathcal{M}_x)$ the word $\Phi(D+v, \partial\Theta_{\mathcal{M}_x}(D,v)) \in W_Y(D_2+v)$ and

$$\Phi(D+v, \partial\Theta_{\mathcal{M}_x}(D,v))\big|_{\partial D_2+v} = \partial\Theta_{\mathcal{M}_x}(D,v). \tag{9}$$

For each $x \in X$ we define $\phi(x)$ by defining for each $(D,v) \in \mathcal{D}_m(\mathcal{M}_x)$

$$\phi(x)|_{D_2+v} \equiv \Phi(D+v, \partial\Theta_{\mathcal{M}_x}(D,v)). \tag{10}$$

If we can establish the following claim, then we will have proven Theorem 2.8.

Claim 9.1. *For each $x \in X$, $\phi(x)$ is a well-defined element of Y and the map $\phi : X \to Y$ is continuous and shift commuting.*

Proof of 9.1. To show $\phi(x)$ is well-defined, observe for each $(D,v) \in \mathcal{D}_m(\mathcal{M}_x)$ we have

$$\begin{aligned} \phi(x)\big|_{\partial D_2+v} &= \Phi(D+v, \partial\Theta_{\mathcal{M}_x}(D,v))\big|_{\partial D_2+v} \\ &= \partial\Theta_{\mathcal{M}_x}(D,v) \\ &= \Theta(\mathcal{M}_x)\big|_{\partial D_2+v}. \end{aligned}$$

Because, for distinct $(D,v),(D',v') \in \mathcal{D}_m(\mathcal{M}_x)$ we have that $(D_2+v)\cap(D'_2+v') \subset (DG(\mathcal{M}_x))_2$ and because $\Theta(\mathcal{M}_x)$ is a well-defined word on $(DG(\mathcal{M}_x))_2$ it follows that $\phi(x)|_{D_2+v}$ and $\phi(x)|_{D'_2+v'}$ agree on the intersection $(D'_2+v')\cap(D_2+v)$, which is to say, $\phi(x)$ is a well-defined point.

For $v,v' \in \mathbb{Z}^2$, let us say the pair (v,v') are *adjacent* if $v' = v+(0,1)$ or $v' = v+(1,0)$. Observe, every adjacent pair of vertices (v,v') is contained in the set D_2+v for some $(D,v) \in \mathcal{D}_m(\mathcal{M}_x)$. Because Y is a matrix SFT and because $\phi(x)\big|_{D_2+v} \in W_Y(D_2)$, it follows that $\phi(x) \in Y$.

Now we wish to show ϕ is shift commuting, *i.e.* $\phi(\sigma^v x) = \sigma^v\phi(x)$. This is probably clear from the construction, nonetheless, let us check it carefully. First we need to check that our definition (Equation 7) of $\partial\Theta_{\mathcal{M}}(D,u)$ is shift commuting.

From Lemma 5.1, $\sigma^v \mathcal{D}_m(\mathcal{M}) = \mathcal{D}_m(\sigma^v \mathcal{M})$; so $(D, u) \in \mathcal{D}_m(\mathcal{M})$ if and only if $\sigma^v(D, u) = (D, u - v) \in \mathcal{D}_m(\sigma^v \mathcal{M})$. Thus the expression $\partial\Theta_{\sigma^v\mathcal{M}}(\sigma^v(D, u))$ is meaningful if and only if $\partial\Theta_{\mathcal{M}}(D, u)$ is. Because $\partial\Theta_{\mathcal{M}}(D, u) \in W_Y(\partial D_2 + u)$ it follows that $\sigma^v\partial\Theta_{\mathcal{M}}(D, u) \in W_Y(\partial D_2 + u - v)$. At the same time $\partial\Theta_{\sigma^v\mathcal{M}}(\sigma^v(D, u))$ is also an element of $W_Y(\partial D_2 + u - v)$ and we claim they are the same element.

To see this claim, observe (by Theorem 7.2) $\sigma^v\Theta(\mathcal{M}) = \Theta(\sigma^v\mathcal{M})$. Thus,

$$\begin{aligned}
\sigma^v(\partial\Theta_{\mathcal{M}}(D, u)) = \sigma^v(\Theta(\mathcal{M})|_{\partial D_2 + u}) &= \sigma^v\Theta(\mathcal{M})|_{\sigma^v(\partial D_2 + u)} \\
&= \Theta(\sigma^v\mathcal{M})|_{(\partial D_2 + u - v)} \\
&= \partial\Theta_{\sigma^v\mathcal{M}}(D, u - v) = \partial\Theta_{\sigma^v\mathcal{M}}(\sigma^v(D, u)).
\end{aligned}$$

The shift commutativity of ϕ is established with the following calculation (noting $\sigma^v\mathcal{M}_x = \mathcal{M}_{\sigma^v x}$).

$$\begin{aligned}
(\sigma^{-v}\phi(\sigma^v x))|_{D_2 + u} = \sigma^{-v}(\phi(\sigma^v x)|_{D_2 + u - v}) &\equiv \sigma^{-v}\Phi(D + u - v, \partial\Theta_{\mathcal{M}_{\sigma^v x}}(D, u - v)) \\
&= \sigma^{-v}\Phi(\sigma^v(D + u), \sigma^v(\partial\Theta_{\mathcal{M}_x}(D, u))) \\
&= \Phi(D + u, \partial\Theta_{\mathcal{M}_x}(D, u)) \\
&\equiv \phi(x)|_{D_2 + u}
\end{aligned}$$

or $\phi(\sigma^v x) = \sigma^v\phi(x)$.

We now address the continuity of ϕ. We know $D \in \mathfrak{D}_m$ implies $D \subset \Lambda_{\kappa m}$. By the continuity of the map $x \mapsto \Theta(\mathcal{M}_x)$, given k there exists $k_2 > 0$ such that if $x|_{\Lambda_{k_2}} = y|_{\Lambda_{k_2}}$, then $\Theta(\mathcal{M}_x)$ and $\Theta(\mathcal{M}_y)$ agree on $\Lambda_{k+2\kappa m+2}$. By the continuity of the map $x \mapsto \mathcal{D}_m(\mathcal{M}_x)$ we know there exists k_3 such that if $x|_{\Lambda_{k_3}} = y|_{\Lambda_{k_3}}$ then $\mathcal{D}_m(\mathcal{M}_x)$ and $\mathcal{D}_m(\mathcal{M}_y)$ agree on $\mathfrak{D}_m \times \Lambda_{k+\kappa m}$. Hence for $k_4 \geq \max\{k_2, k_3\}$ and for x and y with $x|_{\Lambda_{k_4}} = y|_{\Lambda_{k_4}}$ and $(D, v) \in \mathcal{D}_m(\mathcal{M}_x)|_{\mathfrak{D}_m \times \Lambda_{k+\kappa m}} = \mathcal{D}_m(\mathcal{M}_y)|_{\mathfrak{D}_m \times \Lambda_{k+\kappa m}}$ we have that

$$\partial\Theta_{\mathcal{M}_x}(D, v) = \Theta(\mathcal{M}_x)|_{\partial D_2 + v} = \Theta(\mathcal{M}_y)|_{\partial D_2 + v} = \partial\Theta_{\mathcal{M}_y}(D, v)$$

and hence that

$$\phi(x)|_{D_2 + v} = \Phi(D + v, \partial\Theta_{\mathcal{M}_x}(D, v)) = \Phi(D + v, \partial\Theta_{\mathcal{M}_y}(D, v)) = \phi(y)|_{D_2 + v}.$$

Since the union of $D+v$ for $(D,v)\in\mathcal{D}_m(\mathcal{M}_x)\big|_{\mathfrak{D}_m\times\Lambda_{k+\kappa m}}$ covers Λ_k we can conclude $\phi(x)$ and $\phi(y)$ agree on Λ_k. This completes the proof of Claim 9.1 and thus of Theorem 2.8. ■

10. Z^d Prospects for $d>2$

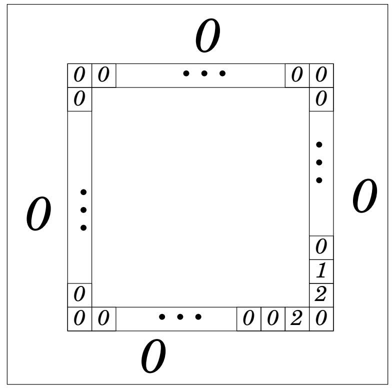
It seems a very reasonable conjecture that the same theorem (2.8) holds for $\mathbb{Z}^d$ systems for $d\geq 3$. While the techniques used to prove Theorem 2.12 work naturally on $\mathbb{Z}^d$ systems for $d\geq 3$, the techniques used in the homomorphism construction (for Theorem 2.8) do not work naturally for $d\geq 3$. Specifically, Proposition 6.3 is false for $d\geq 3$. Nonetheless, I think that by modifying the coding in Sections 7 and 8 the general argument can be extended to $d\geq 3$.

11. Examples

Example 1. In the definition of square filling, for $j\geq 0$, if $u\big|_{\Lambda_K\setminus\Lambda_{K-k+j}}=w\big|_{\Lambda_K\setminus\Lambda_{K-k+j}}$ then Y is said to be *k line, j loss* square filling. A nice situation would be a $\mathbb{Z}^2$ SFT Y presented by matrices with 1 line, 0 loss square filling. That is, for K suitably large ($\geq l$), any locally allowed word on the square annulus $\Lambda_K\setminus\Lambda_{K-1}$, could simply be filled in. One might hope that any positive loss square filling SFT could be recoded by a higher block coding to have zero loss square filling. However, here we construct a 2 line, 1 loss square filling SFT which cannot be recoded by a higher block recoding to a k line 0 loss square filling SFT for any $k\geq 0$.

Let X be the SFT subshift of $\{0,1,2\}^{\mathbb{Z}^2}$ which is formed by dis-allowing the blocks $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{2}$ and $\begin{array}{c}\boxed{0}\\\boxed{2}\end{array}$. For $n>k$ consider the word on the annulus $\Lambda_n\setminus\Lambda_k$ which is all 0s except for two 2s and one 1. See Figure 14. Because of the presence of the word $\begin{array}{cc} & \boxed{2}\\\boxed{2}&\boxed{0}\end{array}$ the annulus cannot be filled without removing one of the 2s. It is easy to construct an algorithm whereby it is sufficient to replace the upper 2 with a 0 and then to fill in the annulus with 1s forming an X allowed word w on the square Λ_n. The word on the annulus will agree with w everywhere on the annulus except at the 2 which was changed to a 0. Moreover, the algorithm may be extended to fill in any X allowed word on $\Lambda_n\setminus\Lambda_{n-k}$ at the expense of replacing 2s with 0s in at most one column in the annulus. Thus one can conclude X is a 2 line 1 loss square filling SFT. The word on the annulus in Figure 14 can be constructed for k,n, and $n-k$

FIGURE 14. A Word on an Annulus Which Cannot Be Recoded to 0-Loss.



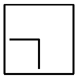
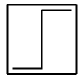
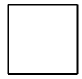
arbitrarily large, but it can never just simply be filled in a 0 loss fashion. Because a higher block recoding amounts to looking at larger blocks it follows that X cannot be recoded with a higher block coding to a k' line 0 loss square filling SFT for any $k' \geq 1$. •

Example 2. This example addresses the appropriateness of the square filling definition. For a $[0,\hat{k}]^2$ scaled SFT it is important that $\hat{k} \leq k \leq l$ because the point of square filling is to allow a local constructibility. An example of local constructibility and how it can fail if $k < \hat{k}$ is the following: Suppose Y is a $[0,\hat{k}]^2$ scaled $\mathbb{Z}^2$ SFT with symbol set S and suppose we had a " square filling" condition where, for example, $k = 1 < \hat{k}$. Suppose w is a word in $W_Y(\mathbb{Z}^2 \backslash \Lambda_n)$ for some $n > 0$, then for any $N > n$ we have $w|_{\Lambda_N \backslash \Lambda_n} \in W_Y(\Lambda_N \backslash \Lambda_n)$ and thus (by square filling) we conclude there exists $W \in W_Y(\Lambda_N)$ such that $W|_{\Lambda_N \backslash \Lambda_{N-1}} = w|_{\Lambda_N \backslash \Lambda_{N-1}}$. We would now have the following unfortunate situation. We have a point $y \in S^{\mathbb{Z}^2}$ such that

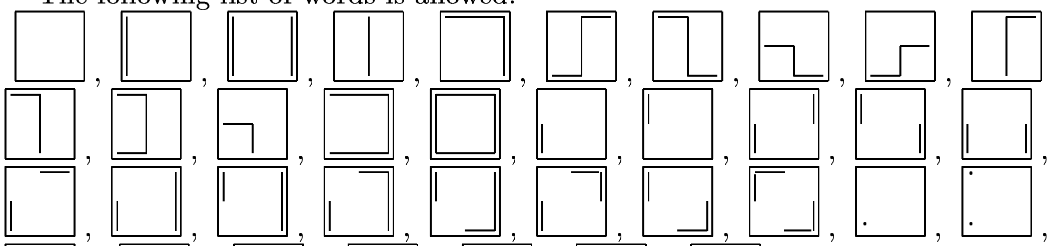
$$y|_{\mathbb{Z}^2 \backslash \Lambda_{N-1}} = w|_{\mathbb{Z}^2 \backslash \Lambda_{N-1}} \in W_Y(\mathbb{Z}^2 \backslash \Lambda_{N-1}) \text{ and } y|_{\Lambda_N} = W \in W_Y(\Lambda_N)$$

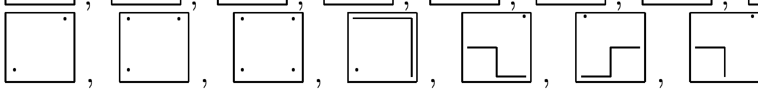
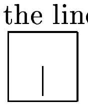
but we could not conclude $y \in Y$ because there exists $v \in \mathbb{Z}^2$ for which $[0,\hat{k}] + v$ is not a subset of either $\mathbb{Z}^2 \backslash \Lambda_{N-1}$ or Λ_N (since $\hat{k} > 1$). Thus we cannot say $y|_{[0,\hat{k}]+v} \in W_Y([0,\hat{k}])$ and hence cannot conclude $y \in Y$. This is a situation we wish to avoid. Thus, in the definition of square filling for a $[0,\hat{k}]$ scaled SFT, we ask that $\hat{k} \leq k \leq l$. •

Example 3. The Closed Curve SFT. Here we present a square mixing SFT which is not a square filling mixing SFT. We describe an SFT $X \subset \{0,1\}^{\mathbb{Z}^2}$ by listing the allowed words on 3×3 squares. Because this is a lot of words we will code them using pictures, for example

 $= \begin{matrix} 0\,0\,0 \\ 1\,1\,0 \\ 0\,1\,0 \end{matrix}$ and  $= \begin{matrix} 0\,1\,1 \\ 0\,1\,0 \\ 1\,1\,0 \end{matrix}$ and  $= \begin{matrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{matrix}$.

The following list of words is allowed.



 , plus a few others presenting isolated 1s (dots), plus all their 90° rotations. The salient feature of all these words is that they code a space of lines. Each point $x \in X$ can be viewed as a collection of lines in $\mathbb{R}^2$ where the lines have the property that they do not branch, cross, or end. For example, the block  , which corresponds to a line ending, is not allowed. (The blocks containing dots are coding portions of corners.)

The SFT X is square mixing for the following reason. For any $x \in X$, if we view $\Lambda_k \subset \mathbb{R}^2$, then the "lines in x" must cross $\partial\Lambda_k$ an even number of times (since lines don't end or merge, each line entering Λ_k must exit). So let $x, y \in X$ and examine $x|_{\Lambda_k}$ and $y|_{\Lambda_{k+15}}$. There are an even number of lines in x impinging upon the set $\partial\Lambda_k$. Connect adjacent line endings as, for example, depicted in Figure 15a . This can be done because there are an even number of such line endings. Do a similar connection process for the lines in y which end on $\partial\Lambda_{k+15}$. Construct z by setting $z|_{\Lambda_{k+5}}$ =(x after connections)$|_{\Lambda_{k+5}}$ and $z|_{\mathbb{Z}^2\setminus\Lambda_{k+9}}$ =(y after connections)$|_{\mathbb{Z}^2\setminus\Lambda_{k+9}}$ and $z_j = 0$ where not already specified. The point $z \in X$, agrees with x on Λ_k and agrees with y on $\mathbb{Z}^2\setminus\Lambda_{k+15}$. Thus X is a square mixing SFT. On the other hand Figure 15b is a locally allowed word which cannot be square filled. Thus X is not square filling. •

Figure 15a. Figure 15b.

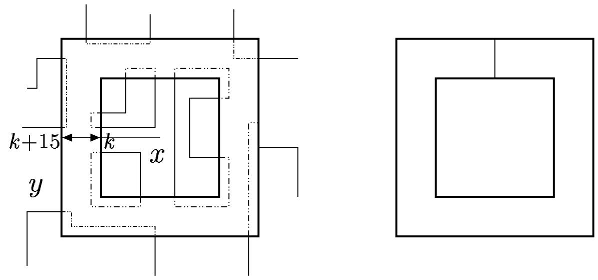


Figure 15. Square Mixing but not Square Filling

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