

# Graphing !!!

These are the steps you should use to graph a function.

1. Look at the function to see if it is the shift or a multiple of something we already know. e.g. You should look at  $(x - 1)^2$  and think of it as  $x^2$  shifted right by 1.
2. Find the intercepts:
  - Find the zeros of the function, these are the x-intercepts.
  - Plug 0 into the function to find the y-intercept.
3. Find the asymptotes, if there are any.
  - Vertical asymptotes, generally where the denominator is zero.
  - Horizontal or oblique asymptote, what the function does as the values of x increase.
4. Compute the first and second derivative, find all Critical Points and Possible Inflection Points.
  - Critical Point, where the first derivative is zero or undefined.
  - Possible Inflection Point, where the second derivative is zero or undefined.
5. Plug the Critical and Possible Inflection Points into the original function and plot them.
6. Investigate the Critical and Possible Inflection Points.
  - First derivative  $> 0$ , function is increasing.
  - First derivative  $< 0$ , function is decreasing.
  - Second derivative  $> 0$ , function is concave up.
  - Second derivative  $< 0$ , function is concave down.
7. Put all this information together to produce the graph of the function.

1. Let's look at the function

$$f(x) = x^3 - 3x^2 - 9x.$$

This is not the shift of any function we know so we need to proceed with the other steps.

2. Factoring  $f(x)$  we get

$$f(x) = x(x^2 - 3x - 9),$$

and the roots of  $x^2 - 3x - 9$  are  $1.5 \pm 0.5\sqrt{45}$ . Therefore, the roots of the polynomial are at  $x = 0, 1.5 + 0.5\sqrt{45}, 1.5 - 0.5\sqrt{45}$ , and the y-intercept is  $y = 0$ .

3. Since this is a polynomial there are no vertical or horizontal asymptotes.

4. Now take the first and second derivatives,

$$\begin{aligned} f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x - 3)(x + 1) \end{aligned}$$

$$\begin{aligned} f''(x) &= 6x - 6 \\ &= 6(x - 1) \end{aligned}$$

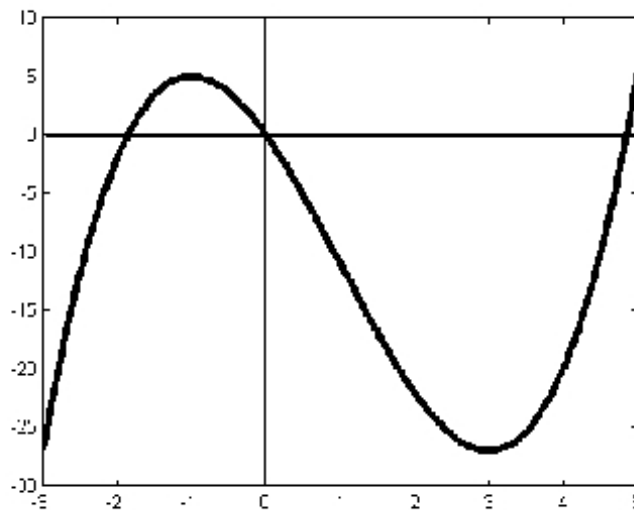
So, the Critical Points are at  $x = 3, -1$  and the Possible Inflection Point is at  $x = 1$

5. Plot the important points,  $f(3) = -27$ ,  $f(-1) = 5$ , and  $f(1) = -11$ .

6. Now we test the important points.

$f(x)$	-	$1.5 - 0.5\sqrt{45}$	+	0	-	$1.5 + 0.5\sqrt{45}$	+
$f'(x)$	↗	↘	↘	↗	↘	↗	↗
$f''(x)$	∪	∩	∩	∪	∪	∩	∩

7. Finally we put all the information together that we have found to get the graph of  $f(x)$ .



1. Let's look at the function

$$f(x) = \frac{x}{x^2-1}.$$

This is not the shift of any function we know so we need to proceed with the other steps.

2. Factoring  $f(x)$  we get

$$f(x) = \frac{x}{(x-1)(x+1)},$$

therefore, the root of the rational function is at  $x = 0$ , and the y-intercept is  $y = 0$ .

3. There are vertical asymptotes at  $x = \pm 1$ , and the horizontal asymptote is  $y = 0$ .

4. Now take the first and second derivatives,

$$\begin{aligned} f'(x) &= \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} & f''(x) &= -1 \cdot \frac{(x^2-1)^2(2x)-(x^2+1)2(x^2-1)(2x)}{(x^2-1)^4} \\ &= \frac{x^2-1-2x^2}{(x^2-1)^2} & &= -1 \cdot \frac{2x(x^2-1)(x^2-1-2x^2-2)}{(x^2-1)^4} \\ &= -1 \cdot \frac{x^2+1}{(x^2-1)^2} & &= -1 \cdot \frac{2x(-x^2-3)}{(x^2-1)^3} \\ & & &= \frac{2x(x^2+3)}{(x^2-1)^3} \end{aligned}$$

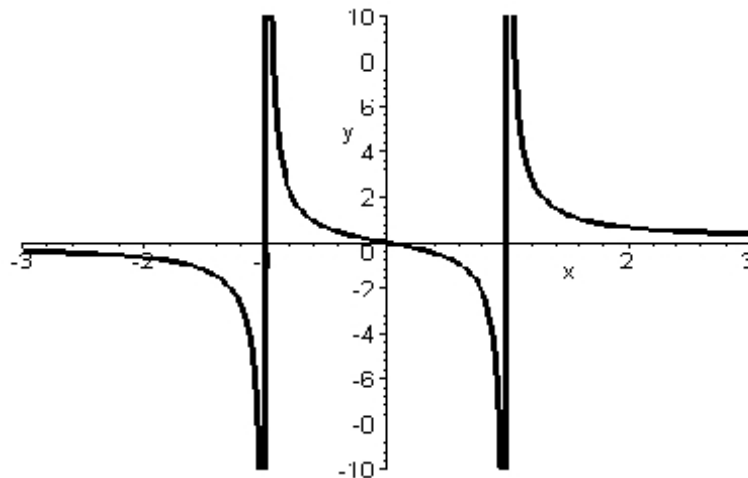
So, the Critical Points are at  $x = \pm 1$  and the Possible Inflection Points are at  $x = \pm 1, 0$ .

5. Plot the important points, that we can,  $f(0) = 0$ .

6. Now we test the important points.

$f(x)$	—	-1	+	0	-	1	+
$f'(x)$	↘	-1		↘		1	↘
$f''(x)$	∩	-1	∪	0	∩	1	∪

7. Finally we put all the information together that we have found to get the graph of  $f(x)$ .



1. Let's look at the function

$$f(x) = \frac{x^2-4}{5x-15}.$$

This is not the shift of any function we know so we need to proceed with the other steps.

2. Factoring the numerator and denominator of  $f(x)$  we get

$$f(x) = \frac{(x-2)(x+2)}{5(x-3)}.$$

So the x-intercepts are at  $x = 2$  and  $x = -2$ , and the y-intercept is at  $y = \frac{4}{15}$ .

3. There is a vertical asymptote when  $x=3$ , and using polynomial long division we get

$$f(x) = \frac{1}{5}x + \frac{3}{5} + \frac{5}{5x-15}$$

so as  $x$  becomes large  $f(x)$  approaches the line  $y = \frac{1}{5}x + \frac{3}{5}$ , this is called an oblique asymptote.

4. Now take the first and second derivatives,

$$\begin{aligned} f'(x) &= \frac{(5x-15)(2x)-(x^2-4)(5)}{(5x-15)^2} & f''(x) &= \frac{1}{5} \frac{(x-3)^2(2x-6)-(x^2-6x+4)(2)(x-3)(1)}{(x-3)^4} \\ &= \frac{5(2x^2-6x-x^2+4)}{5^2(x-3)^2} & &= \frac{1}{5} \frac{(2)(x-3)((x-3)^2-(x^2-6x+4))}{(x-3)^4} \\ &= \frac{1}{5} \frac{x^2-6x+4}{(x-3)^2} & &= \frac{1}{5} \frac{(2)(x^2-6x+9-x^2+6x-4)}{(x-3)^3} \\ & & &= \frac{2}{(x-3)^3} \end{aligned}$$

So  $x = 3$  is both a critical point and a possible inflection point. The other critical points are the zeros of  $x^2 - 6x + 4$  which are  $x = 3 \pm \sqrt{5}$ .

5.  $f(3 - \sqrt{5}) = \frac{10-6\sqrt{5}}{-5\sqrt{5}} \approx 0.3055728$  and  $f(3 + \sqrt{5}) = \frac{10+6\sqrt{5}}{5\sqrt{5}} \approx 2.0944271$ .

6. Now we test the important points.

$f(x)$	-	-2		+		2	-	3	+		
$f'(x)$			↗	$3 - \sqrt{5}$	↘			3	↘	$3 + \sqrt{5}$	↗
$f''(x)$			⌒		3			3	⌒		

7. Finally we put all the information together that we have found to get the graph of  $f(x)$ .

