Partial Fraction Decomposition

In order to add two rational functions you first find a common denominator and then combine and simplify the numerator.

$$\begin{aligned}\frac{x^2}{x^3+1}+\frac{1}{x^2+1} &= \frac{x^2(x^2+1)+1(x^3+1)}{(x^3+1)(x^2+1)}\\ &= \frac{x^4+x^3+x^2+1}{x^5+x^3+x^2+1}\end{aligned}$$

The idea behind *partial fraction decomposition* is to undo this process. We do three mayor types of problems in this class:

1. Non-repeated linear factors, e.g. $\frac{1}{(x-1)(x+2)}$.
2. Repeated linear factors, e.g. $\frac{1}{(x-1)(x+2)^2}$.
3. Irreducible quadratic factors, e.g. $\frac{1}{(x-1)(x^2+1)}$.

Below is an example of each type of decomposition. Remember, when you go to do other problems, that the degree of the numerator must be less than the degree of the denominator.

Example 1

To find the integral $\int \frac{1}{(x-1)(x+2)}\,dx$ we start by decomposing the fraction.

$$\begin{aligned}\frac{1}{(x-1)(x+2)} &= \frac{A}{x-1}+\frac{B}{x+2}\\ &= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}\end{aligned}$$

Therefore,

$$A(x+2)+B(x-1)=1.$$

Now, if $x=1$ then we get

$$3A=1 \text{ and } A=\frac{1}{3},$$

and if $x=-2$ then we get

$$-3B=1 \text{ and } B=\frac{-1}{3}.$$

So,

$$\begin{aligned}\int \frac{1}{(x-1)(x+2)}\,dx &= \frac{1}{3}\int \frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{1}{x+2}\,dx\\ &= \frac{1}{3}\ln(|x-1|)-\frac{1}{3}\ln(|x+2|)+c.\end{aligned}$$

Example 2

Find the integral $\int \frac{x^2}{(x-1)(x-2)^2}$. Again we start by decomposing the fraction.

$$\begin{aligned}\frac{x^2}{(x-1)(x-2)^2} &= \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ &= \frac{A(x-2)^2+B(x-1)(x-2)+C(x-1)}{(x-1)(x-2)^2}\end{aligned}$$

Therefore,

$$A(x-2)^2 + B(x-1)(x-2) + C(x-1) = x^2.$$

Now, if $x = 2$ then we get

$$C = 4,$$

if $x = 1$ then

$$A = 1,$$

and if $x = 0$ then

$$4 + 2B - 4 = 0 \text{ and } B = 0.$$

So,

$$\begin{aligned} \int \frac{x^2}{(x-1)(x-2)^2} &= \int \frac{1}{x-1}\, dx + 4\int \frac{1}{(x-2)^2}\, dx \\ &= \ln(|x-1|) - \frac{4}{x-2} + c. \end{aligned}$$

Example 3

Find the integral $\int \frac{x}{(x-1)(x^2+1)}\, dx$. First we decompose the fraction.

$$\begin{aligned} \frac{x}{(x-1)(x^2+1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\ &= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)} \end{aligned}$$

Therefore,

$$A(x^2+1) + (Bx+C)(x-1) = x.$$

Now, if $x = 1$ then

$$2A = 1 \text{ and } A = \tfrac{1}{2},$$

if $x = 0$ then

$$\tfrac{1}{2} - C = 0 \text{ and } C = \tfrac{1}{2},$$

and if $x = 2$ then

$$\tfrac{5}{2} + 2B + \tfrac{1}{2} = 2 \text{ and } B = \tfrac{-1}{2}.$$

So,

$$\begin{aligned} \int \frac{x}{(x-1)(x^2+1)}\, dx &= \frac{1}{2}\int \frac{1}{x-1}\, dx + \frac{1}{2}\int \frac{1-x}{x^2+1}\, dx \\ &= \frac{1}{2}\int \frac{1}{x-1}\, dx + \frac{1}{2}\int \frac{1}{x^2+1}\, dx - \frac{1}{2}\int \frac{x}{x^2+1}\, dx \\ &= \frac{1}{2}\ln(|x-1|) + \frac{1}{2}\arctan(x) - \frac{1}{4}\ln(|x^2+1|) + c \end{aligned}$$

Trigonometric Substitution

Trigonometric Substitution, trig sub, uses trig identities to simplify integrals and make them easier to manage. In particular you use:

1. $a^2 - a^2 \sin^2(\theta) = a^2 \cos^2(\theta)$
2. $a^2 + a^2 \tan^2(\theta) = a^2 \sec^2(\theta)$
3. $a^2 \sec^2(\theta) - a^2 = a^2 \tan^2(\theta)$

To solve problems using trig sub you use one of three possible substitutions:

1. if you have an integral with a $a^2 - u^2$ then use $u = a\sin(\theta)$,
2. if you have an integral with a $u^2 + a^2$ then use $u = a\tan(\theta)$,
3. if you have an integral with a $u^2 - a^2$ then use $u = a\sec(\theta)$.

Find the integral of $\int \frac{1}{\sqrt{1-x^2}}\,dx$. Since we have something that looks like $a^2 - u^2$ we will use the substitution $u = a\sin(\theta)$.

$$\begin{aligned} x &= \sin(\theta) \\ dx &= \cos(\theta)\,d\theta \\ 1 - x^2 &= 1 - \sin^2(\theta) = \cos^2(\theta) \end{aligned}$$

And, we integrate,

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)}}\,d\theta \\ &= \int \frac{\cos(\theta)}{\cos(\theta)}\,d\theta \\ &= \int d\theta \\ &= \theta + c \\ &= \arcsin(\theta) + c \end{aligned}$$

Question: The function $\sin(\theta)$ is only defined between 1 and -1, however x may be defined everywhere, so how can we make the substitution $x = \sin(\theta)$?