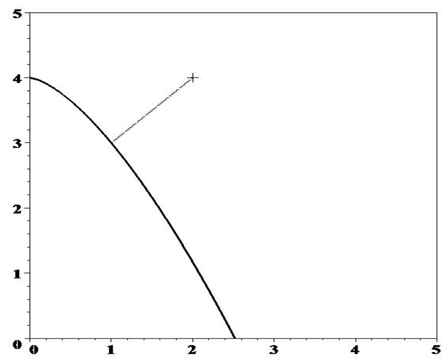
Optimization Examples

Steps to Optimization:

- Read the problem.
- Reread the problem.
- Draw a picture or graph if appropriate.
- Identify the given information
  - What are the variables?
  - What are the constants?
  - Are there any constraints?
  - Label the graph or picture.
- What quantity needs to be maximized or minimized?
- Find an appropriate equation for what needs to be maximized or minimized, and reduce it to one variable.
- Find the derivative and critical points for your equation.
- Test your critical points and end points (where appropriate).
- Reread the question and make sure you have answered what was asked.

1. Given the function $f(x) = 4 - x^{3/2}$ find the point on $f(x)$ which minimizes the square of the distance between the function and the point $(2, 4)$

   (a) Graph the function and point:



   (b) Given Information: We need to minimize the distance from $(2, 4)$ to the curve and each point on the curve looks like $(x, f(x)) = (x, 4 - x^{3/2})$.

   (c) Equation to Optimize:

$$Distance^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2$$

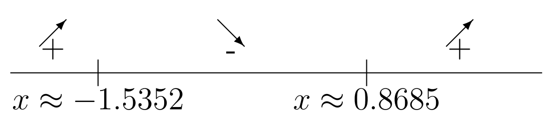
   (d) So using the distance formula and the given points we get

$$Distance^2 = (x - 2)^2 + (4 - x^{3/2} - 4)^2 = (x - 2)^2 + x^3.$$

   Then to minimize the distance squared we first take the derivative and find the critical points:

$$(Distance^2)' = 2(x - 2)(1) + 3x^2 = 3x^2 + 2x - 4 = 0$$

   So the critical points are approximately $x \approx -1.5352$ and $x \approx 0.8685$. Using the first derivative test

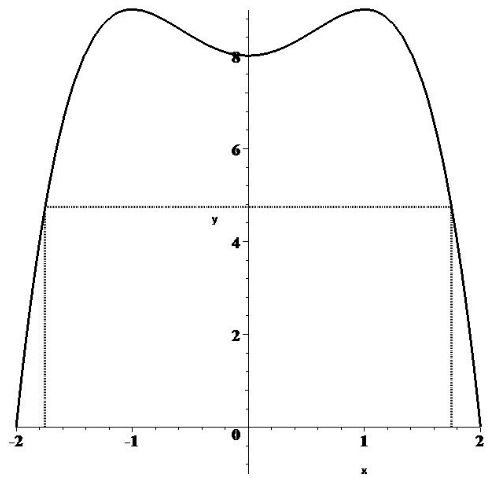


   we see a minimum at about $x \approx 0.8685$. Thus the point on $f(x) = 4 - x^{3/2}$ closest to $(2, 4)$ is approximately $(0.8685, 3.1906)$ or exactly

$$\left(\frac{-1+\sqrt{13}}{3}, f\left(\frac{-1+\sqrt{13}}{3}\right)\right).$$

2. Inscribe a rectangle between $y = -x^4 + 2x^2 + 8$ and the x-axis. What values of $-2 < x < 2$ will maximize the area of the rectangle.

   (a) Graph:



   (b) Given Information: $f(x) = -x^4 + 2x^2 + 8$ with $-2 < x < 2$.

   (c) Equation to Optimize:

$$Area = Width \cdot Height = 2x \cdot f(x) = -2x^5 + 4x^3 + 16x.$$

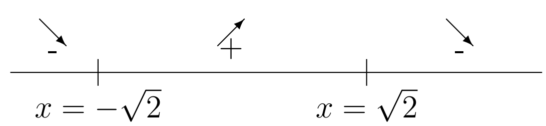
   (d) The derivative of area in this case is

$$A' = -10x^4 + 12x^2 + 16$$

   This factors into

$$A' = -2(5x^2 + 4)(x^2 - 2) = -2(x - \sqrt{2})(x + \sqrt{2})(5x^2 + 4)$$

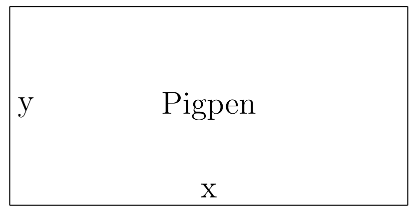
   so the critical points, where the derivative is 0, are $x = \pm 2$. Using the first derivative test



   we see that the area is maximized at $x = \sqrt{2}$.(Note that it says the area is minimized when $x = -\sqrt{2}$ but this really doesn't make sense because we are looking for the dimensions of a rectangle and because $f(x)$ is symmetric about the y-axis, so we ignore the $x = -\sqrt{2}$ critical point.)

3. A farmer wants to construct a rectangular pigpen using $1000ft$ of fencing. What dimensions should should the pigpen be to maximize area and what is the maximum area?

   (a) Diagram:



   (b) Given Information:

$$Perimeter = 2x + 2y = 1000$$

   (c) Equation to Optimize:

$$Area = xy$$

   (d) Since

$$2x + 2y = 1000$$

   we can solve for y and get

$$y = 500 - x.$$

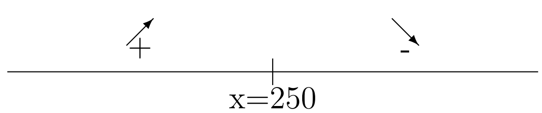
   Then we can rewrite area as

$$A = x(500 - x) = 500x - x^2.$$

   Now we find where the maximum value of the area is by taking the derivative and finding the critical point(s)

$$A' = 500 - 2x = 0$$

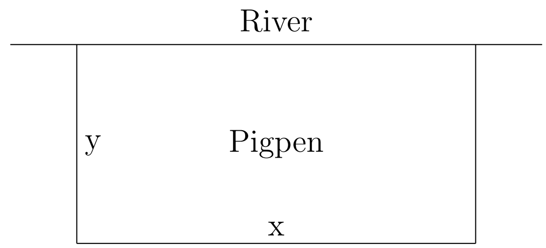
   which in this case is at $x = 250ft$. We make sure this is the maximum value by using the first derivative test:



   Thus, since $y = 500 - x$, the dimensions that will maximize the area are $x = 250ft$ and $y = 250ft$ so that the maximum area is $62500sqft$.

4. Now suppose that the farmer wants to build his pigpen along a river (the pigs are fond of swimming and it gets their smell a little further from the farm house). He still has $1000ft$ but he also has the river to build along, so what dimensions will now maximize the area assuming three sides of the encloser are fence and one is river.

   (a) Diagram:



   (b) Given Information:

$$Perimeter = x + 2y = 1000$$

   (c) Equation to Optimize:

$$Area = xy$$

   (d) Since

$$x + 2y = 1000$$

we can solve for y and get

$$y = 500 - \tfrac{1}{2}x.$$

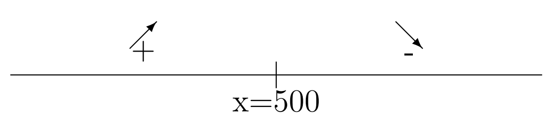
Then we can rewrite area as

$$A = x(500 - \tfrac{1}{2}x) = 500x - \tfrac{1}{2}x^2.$$

Now we find where the maximum value of the area is by taking the derivative and finding the critical point(s)

$$A' = 500 - x = 0$$

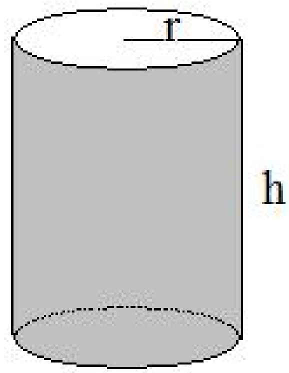
which in this case is at $x = 500ft$. We make sure this is the maximum value by using the first derivative test:



Thus, since $y = 500 - \frac{1}{2}x$, the dimensions that will maximize the area are $x = 500ft$ and $y = 250ft$ so that the maximum area is $125000sqft$.

5. An aluminium manufacturing company is working on designs for soup cans. In one design they are going to use the same gauge metal for the whole can, it costs \$0.002 per square centimeter of aluminum. If the can must hold 430ml of soup what dimensions will minimize the cost of the can. (1ml=1cc)

(a) Diagram:



(b) Given Information:

- $Volume = \pi r^2 h = 430ml$
- $Cost/cm^2 = \$0.002$
- $1ml = 1cc$

(c) Equation to Optimize:

$$Cost\ of\ Can = 0.002 * Surface\ Area = 0.002(2\pi r^2 + 2\pi r h)$$

(d) Since $V = \pi r^2 h = 430$ we can write $h = \frac{430}{\pi r^2}$. So the cost can be given as

$$C = 0.002(2\pi r^2 + 2\pi r \frac{430}{\pi r^2}$$

which simplifies to

$$C = 0.004(\pi r^2 + 430r^{-1}).$$

Taking the derivative we get

$$C' = 0.004(2\pi r - 430r^{-2}).$$

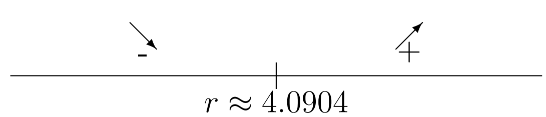
Setting this equal to 0 and solving for r we get

$$r^3 = \frac{430}{2\pi}$$

or

$$r \approx 4.0904.$$

We make sure this is the minimum value by using the first derivative test:



So the cost of the can is minimized when $r \approx 4.0904\,cm$ and $h = \frac{430}{\pi r^2} \approx 8.1806\,cm$. (Note that the height is equal to twice the radius, or the diameter, so from the side the can looks square.)

6. After a number of class action law suits involving exploding cans of soup the aluminium company decides to revise its can design. It now wants to use a heavier gauge metal for the top and the bottom of the can, \$0.004 per square centimeter, and the \$0.002 per square centimeter for the sides. Find the new dimensions to minimize the cost of the can with a volume of 430 ml

(a) Given Information:

- $Volume = \pi r^2 h = 430ml$
- $Cost/cm^2 =$ \$0.002 for the sides.
- $Cost/cm^2 =$ \$0.004 for the top and bottom.
- $1ml = 1cc$

(b) Equation to Optimize:

$$Cost\ of\ Can = 0.002 * Surface\ Area\ of\ the\ Sides + 0.004 * Surface\ Area\ of\ the\ top\ and\ bottom$$

which is equal to

$$C = 0.002(\pi r h) + 0.004(2\pi r^2)$$

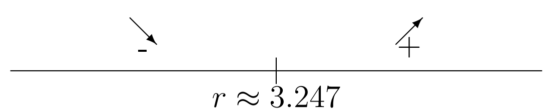
(c) Since $V = \pi r^2 h = 430$ we can write $h = \frac{430}{\pi r^2}$. So the cost can be given as

$$C = 0.002(430r^{-1}) + 0.004(\pi r^2).$$

The derivative is now

$$C' = -0.860r^{-2} + 0.008\pi r.$$

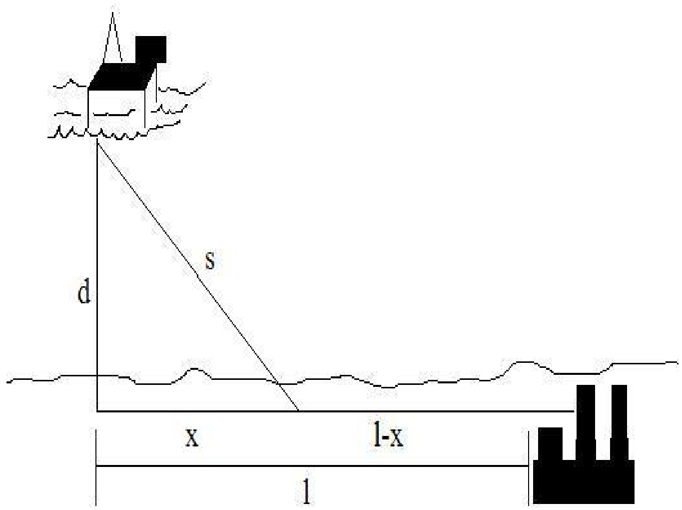
Setting this equal to 0 and solving for r we get $r^3 = \frac{0.860}{0.008\pi}$ so that $r \approx 3.247$. We make sure this is the minimum value by using the first derivative test:



So the cost of the can is minimized when $r \approx 3.247\,cm$ and $h = \frac{430}{\pi r^2} \approx 12.982\,cm$. (Note that the height is now equal to four times the radius, or twice the diameter.)

7. An oil rig is 10 miles out to sea and the refinery that the oil needs to go to is 20 miles up the shore line from the oil rig. If it costs \$25 million per mile to place a pipeline in the water and \$10 million per mile on land where should the pipeline make landfall in order to minimize cost.

   (a) Diagram:



   (b) Given Information:
   - 10 miles to land
   - 20 miles along the shore
   - \$25 million per mile in the sea
   - \$10 million per mile on land

   (c) Equation to Optimize:

$$Cost = 25s + 10(l - x)$$

   (d) Given $d = 10$ and $l = 20$ we can write the cost as

$$C = 25\sqrt{100 + x^2} + 200 - 10x.$$

   Then the derivative will be

$$C' = 25x(100 + x^2)^{-1/2} - 10$$

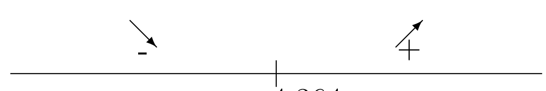
   which when if we set equal to 0 and solve for x gives

$$x^2 = \frac{10000}{525}$$

   or

$$x \approx 4.364.$$

   We make sure this is the minimum value by using the first derivative test:



$$x \approx 4.364$$

   So the cost is minimized when $x \approx 4.364\,miles$.

8. A restaurant has a seating area that can seat up to 25 tables with 4 people per table. If they only put in 15 tables then they can charge on average \$20 per meal and fill the restaurant. However, for each additional table they put in they must decrease the price of the average meal by \$1. assuming all the tables are full, what number of tables will maximize their profit.

   (a) Given Information:
      - They charge \$20 per meal with 15 tables.
      - They must subtract \$1 per meal for each table over 15.
      - They can fit at most 25 tables.
      - Assume all the tables are full, 4 people.

   (b) Equation to Optimize:

$$Profit = 4 \cdot tables \cdot \$(20 - (number\ of\ tables\ over\ 15))$$

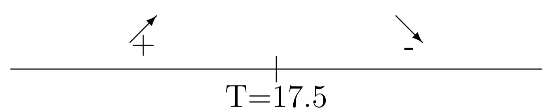
   which we can simplify to

$$P = 4T(20 - (T - 15)) = 140T - 4T^2.$$

   (c) To find the optimum number of tables we first take the derivative and find the critical points

$$P' = 140 - 8T = 0.$$

   So the critical point is at $T = 17.5$ which according to the first derivative test



   does give us a maximum. Thus if the restaurant wants to maximize profit it should have either 17 or 18 tables (assuming that they don't want to cut a table in half).

More Samples:

1. A rectangle has one side on the x-axis, one side on the y-axis, one vertex at the origin, and one vertex on the curve
$$y = e^{-2x}$$
for $x \geq 0$. Find the maximum possible area. Find the minimum possible perimeter.

2. A rectangle is inscribed under the arch of the curve
$$y = 4\cos(0.5x)$$
from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with the largest area, and what is the largest area?

3. Find the point on the curve $y = 0.5\,e^{-x}$ which is closest to the point $(-1, 1)$.

4. Given $x^2 + y^2 = 5$ find the minimum and maximum values of $P = x^2 e^{y^2-5}$.

5. A frictionless cart, attached to the wall by a spring, is pulled 10cm from rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is given by
$$s = 10\cos(\pi t).$$
What is the cart's maximum speed on this time interval.

6. The range R of a projectile fired with an initial velocity of v_0 is given by
$$R = \frac{(v_0 \sin(2\theta))}{g}$$
where g is acceleration due to gravity. Find the angle θ such that the range is maximized.

7. Fifty elk are introduced into a game preserve. It is estimated that their population will increase according to the model
$$p(t) = \frac{250}{1+4e^{-t/3}}$$
where t is measured in years. At what rate is the population increasing when t=2? After how many years is the population increasing most rapidly?

8. A woman pulls a sled which, together with its load, has a mass of m kg. If her arm makes an angle of θ with her body (assumed to be vertical) and the coefficient of friction is $\mu = 0.15$, then the least force, F, she must exert is given by
$$F = \frac{mg\mu}{\sin(\theta)+\mu\cos(\theta)}$$
where g is the constant acceleration of gravity. Find the angle θ between 0 and $\pi/2$ which minimizes F.