Lab 3 – Euler's method

My Goals:

1. To study the relation between a function and its derivative
2. To introduce a method of numerical analysis
3. To introduce Excel as mathematical software
4. To improve skills at writing mathematical lab reports

Begin your lab report with a statement of your own goals for this lab. They should be based in part, but not entirely, on my goals. Some of them are likely to be personal goals.

Introduction:

We recently learned what we called the "differential formula:"

$$f(x+h) \approx f(x) + h \cdot f'(x)$$

Typical example:

You should be very good by now at finding the derivative of the square root function. Your calculations would look like this:

$$f(x) = \sqrt{x} = x^{1/2}$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

Here, $n = 1/2$, and $n - 1 = -1/2$

so

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

We know that $\sqrt{9} = 3$. We can use this to estimate $\sqrt{10}$, as follows:

Take

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Take $x = 9$ so that $f(x) = 3$ and $f'(x) = \frac{1}{2 \cdot 3} = \frac{1}{6}$, and take $h = 1$, so $x + h = 10$.

The differential formula tells us that

$$f(10) = f(9+1) \approx f(9) + 1 \cdot f'(9)$$
$$= 3 + 1 \cdot \frac{1}{6}$$
$$\approx 3.1667$$

Meanwhile, my calculator gives the square root of 10 (to 4 decimal places) as 3.1623, an error of 0.0044, or about 0.14%.

Note: It is important to understand error. Error describes the difference between an actual value and a measured or estimated value. In Chem Lab, you have measured values. Here we are calculating estimated values. Let x_a be the actual value and let x_e be the estimated or measured value.

We measure two kinds of error, absolute error and relative error. They are given by the formulas

$$e_{absolute} = x_e - x_a$$
$$e_{relative} = \frac{e_{absolute}}{x_a}$$

We usually report relative error as a percent.

An estimate that is high has positive error and one that is negative has negative error. Usually, we report the absolute value of the error.

The second part of your lab report should be a summary of the principles on which the lab is based. In this case, it is this differential formula.

This should be a new section of your lab, with an appropriate section heading. (Don't start on a new page though.)

Task 1: Use this procedure to estimate the following values:

1. $\sqrt{2}$ (2 is one more than a perfect square)
2. $\sqrt{3}$ (3 is one less than a perfect square)
3. $\sqrt{8}$ (8 is one less than a perfect square)
4. $\sqrt{48}$ (48 is one less than a perfect square)

This is another new section of your lab report.

Explain your procedure for this step.

Then make a *nice* table that gives the value you are estimating, your estimate (to four decimal places), the actual value (again, to 4 decimal places; be careful about rounding. I check) and your absolute and relative errors.

Some errors are worse than others. Try to explain this. If you can't explain it, say you can't do it instead of trying to bluff your way through it. If you're not sure, give it a try, but say when you are speculating.

Digging a little deeper:

Since 48 = 16 * 3, we know that $\sqrt{48} = \sqrt{16} \cdot \sqrt{3} = 4 \cdot \sqrt{3}$, so we can estimate the square root of 3 by dividing our estimate of $\sqrt{48}$ by 4.

1. Is this more accurate than using $\sqrt{4}$ to estimate $\sqrt{3}$?
2. Can you explain why or why not?
3. Since 8 = 4 * 2, we know $\sqrt{8} = 2\sqrt{2}$. Can you use the same trick to improve your estimate of $\sqrt{2}$?

Answering this part of the problem should be another new section. There are a few calculations, that you should present in a brief but clear way, and some explanation that should be clear and it should show me that you understand what you are doing.

A little more challenging:

Use the same technique on the sine function to estimate sin 50° knowing that the sin 45° is sqrt(2)/2 and that the derivative of the sine function is the cosine function.

This problem is extra tricky because I'm asking a question in degrees, but the derivative of the sine function expects that you are measuring angles in radians, not degrees. You will have to convert between radians and degrees. Also, for this problem, *h* is not equal to 1. Make sure you deal with this.

Show your calculations. You can use the value of π, all the basic arithmetic operations and the square root, but you shouldn't use any sine or cosine calculations other than already knowing sin 45°.

Calculate your errors and write about the experience.

Smaller steps:

Watch me try to estimate $\sqrt{5}$ knowing $\sqrt{4}$ and the derivative of the square root function. Note that $\sqrt{5} = 2.2361$ (to 4 decimal places.)

f(4) = 2
f '(4) = 1/(2*sqrt(2)) = ¼
Take h = 1.
so f (5) = f(4 + 1) ≈ f(4) + 1*f '(4)
= 2 + ¼
= 2.2500

The error is +0.0139, about 0.62%.

Let's try this in two steps; first use f(4) and h = 0.5 to estimate f(4.5), and then use that estimate to estimate f(5). Perhaps two smaller steps will be more accurate than one larger step.

First step:
f(4) = 2, f '(4) = ¼, as above. Take h = 0.5
so f(4.5) = f(4 + 0.5) ≈ f(4) + 0.5*f '(4)
= 2 + 1/8
= 2.1250

Second step:
f(4.5) ≈ 2.1250, and f '(4.5) = 1/(2*sqrt(4.5)) ≈ 1/(2*2.1250) = 0.23529
so f(5.0) = f(4.5 + 0.5) ≈ f(4.5) + 0.5*f '(4.5)
≈ 2.1250 + 0.5 * 0.23529
= 2.2426
The error is 0.0065, about 0.29%, or half the other error.

Smaller steps seem to help.

You could just as well do more work and take 10 steps of lengths 0.1, and perhaps get an even better estimate.

When you take more than one step, this method of using differentials a step at a time is called **Euler's Method**, hence the tile of this lab.

Task 2: ln 2 = 0.6931

Let f(x) = ln x, the natural logarithm function. You know (and if you don't know, your calculator will tell you) that ln (1) = 0. Also, the derivative of ln x is 1/x.

The differential formula, with h = 1, tells us that

$$\ln 2 = \ln(1+1) \approx \ln(1) + 1 \cdot \frac{d}{dx} \ln x_{x=1} = 0 + 1 \cdot \frac{1}{1} = 1$$

So, this estimate that ln 2 = 1 is very poor, with a relative error of over 44%.

Your task is to use more and more steps, shorter and shorter, until you find ln 2 with an error of less than 5%. This means your estimate should roughly be between 0.66 and 0.72.

Hint: If you go as high as 25 steps and your error is still bigger than 5%, you are doing something wrong, and you should try to find your mistake instead of spending your time taking more and more steps.

Hint: The spreadsheet program Excel is likely to help a lot in this part of the lab.

This is the part of the lab we sometimes call the "Capstone." You are required to put together all the pieces that you have learned earlier in the lab. The work is harder, but also it is a bit more difficult to write up. You should be careful to include at least the following as you write up this part of the lab:

1. State the problem. Describe what you are trying to do.
2. Describe your procedure. Tell why you think it should work.
3. Describe your results. Use tables and graphs appropriately.
4. Analyze your results.
5. State your conclusions.

Technical conclusions:

Describe your conclusions about Euler's method and the differential formula.

Over-all conclusions

At the beginning of the lab, you stated some goals for this lab, including, perhaps some personal ones. Conclude the lab with an assessment of how well those goals were satisfied.

Hint: Make your lab look good

Word has a feature, >Insert>Object>Equation. Use this to build nice mathematical equations and embed them in Word documents.

One last hint: How will you be graded? See the potential grading rubric below. It is probably the meter stick by which the quality of your work will be measured.

Potential Grading rubric

Lab 3 – Euler's method

Attractiveness	0	1	2	3	4	5
5 = excellent						
3 = good						
Statement of goals	0	1	2	3	4	5
Description of principles	0	1	2	3	4	5
Calculate square roots including errors	0	1	2	3	4	5
Digging deeper	0	1		2		3
Euler's method						
Statement, procedure	0	1		2		3
Results and table	0	1	2	3	4	5
Graphs and tables	0	1		2		3
Analysis and concl.	0	1		2		3
Technical conclusions	0	1	2	3	4	5
Goals conclusions	0	1		2		3

Total ______/45