

Lab 5 – Area

My Goals:

1. To sharpen derivative skills by finding “anti-derivatives,”
2. To discover techniques for estimating area,
3. To discover the relation between anti-derivatives and area,

Introduction

If you remember when we started the course, we said that Calculus has two main parts. The first deals with slopes of functions, and we understand that very well now that we know so much about derivatives.

The second part of Calculus is about areas. The two are related by something called the Fundamental Theorem of Calculus.

We will start by working with “anti-derivatives.” Given a function, $f(x)$, we will look for a function $F(x)$ that has $f(x)$ as its derivative. This is the “What’s My Function?” section of the Lab, and is covered in Chapter 4 section 8 of your textbook. Then we will relate this to areas. This is the “Area” section of the Lab and is covered in Chapter 5 section 1 of the textbook.

First part – What’s My Function?

Derivatives and Anti-derivatives

If the derivative of $F(x)$ is $f(x)$, then we call $F(x)$ the anti-derivative of $f(x)$. It is usually harder to find anti-derivatives than it is to find derivatives. Sorry. With practice, you learn to recognize patterns and do some tricks.

Polynomials

With polynomials, you just do the polynomial rule backwards. If the derivative is $f(x) = nx^{n-1}$, then the anti-derivative is $F(x) = x^n$. A slightly different, but more useful form of this same rule is that if $f(x) = x^m$, then $F(x) = \frac{1}{m+1}x^{m+1}$.

Here are a few to practice with:

function	its antiderivative	rule you used
$2x$		polynomial rule
$x^2 - 2x + 1$		polynomial rule
\sqrt{x}		polynomial rule
$8x - \frac{2}{x^4}$		polynomial rule

Chain rule tricks

If you apply the chain rule, you always get a result of the form $f'(x)g(f(x))$. That is, you end up multiplying by the derivative of something on the “inside” of another function. For example,

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$$

The $2x$ on the outside is the derivative of the x^2 on the inside. That is a clue that you probably want to apply the chain rule backwards. Here are a few to practice on:

function	its antiderivative	rule you used
$3x^2 e^{(x^3)}$		chain rule
$(\cos x)(\sec^2(\sin x))$		chain rule
$(-\sin x) \frac{1}{\cos x}$		chain rule

Sometimes you have to be a little bit tricky to make some part on the outside exactly equal to the derivative of the inside. Here’s an example:

$$xe^{(x^2)} = \frac{1}{2} \left(2xe^{(x^2)} \right)$$

By multiplying and dividing by 2, it lets us put $2x$, exactly the derivative of x^2 outside the function. Try this with a couple.

function	its antiderivative	rule you used
$x^2 e^{(x^3)}$		chain rule with trick
$x(\sec^2 x^2)$		chain rule with trick

Product rule

You can sometimes tell that the derivative involved the product rule if there is a sum and parts of one term are equal to the derivatives of the other term. An example of this is

$$x \cos x + \sin x$$

The $\cos x$ in the first term is the derivative of the $\sin x$ in the second term, and the unwritten 1 in the second term is the derivative of the x in the first term. So, we can re-write this as

$$x \cdot \frac{d}{dx} \sin x + \sin x \cdot \frac{d}{dx} x$$

We see that this last is what we get if we apply the Leibniz Rule (a.k.a. Product Rule) to the function

$$x \cdot \sin x ,$$

so that's the anti-derivative of $x \cos x + \sin x$.

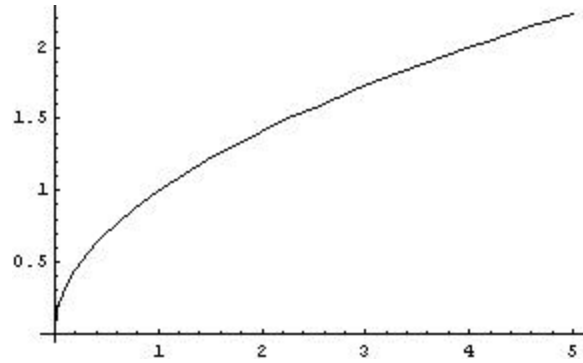
Here are some to practice on

function	its antiderivative	rule you used
$e^x \cdot \sin x + e^x \cdot \cos x$		product rule
$\cos^2 x - \sin^2 x$		product rule
$\frac{e^x}{x} + e^x \ln x$		product rule

Second part – Area

Upper sums, lower sums, and area

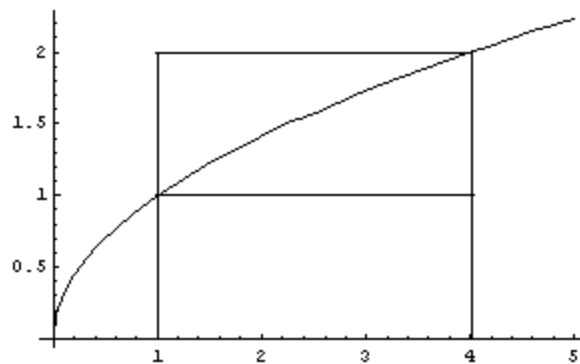
We are going to try to estimate the area beneath the curve $y = \sqrt{x}$ and on the interval $1 \leq x \leq 4$. The function is graphed at the right. You should “doodle” on the picture to show yourself the area we are talking about. Its shape is kind of like a trapezoid, but with a curvy top.



First estimate

We can make a first estimate of the area of the region.

First, we can enclose it in the large rectangle shown at the right. The coordinates of the corners of the rectangle are $(1,0)$, $(4,0)$, $(4,2)$ and $(1,1)$. The base of the rectangle is length 3 and the height is 2, so the area of the rectangle is 6, and so the area under the curve is less than 6.



Then, we draw the smaller rectangle *inside* our area. The coordinates of its corner points are

_____, _____, _____, and _____,

The length of its base is _____, and its height is _____, so its area is _____.

Since this rectangle is entirely inside our area, we conclude our area is greater than 3. If we denote our area by A , we can say that

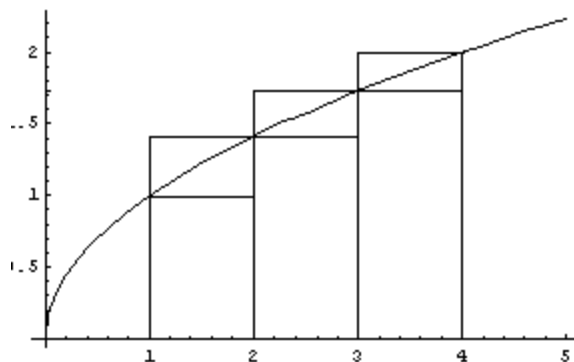
$$\underline{\hspace{2cm}} \leq A \leq \underline{\hspace{2cm}}$$

A better estimate

We can get more accurate estimates by using more and narrower rectangles, as illustrated at the right.

We can take three sub-intervals $[1,2]$, $[2,3]$ and $[3,4]$, and find the maximum value on each sub-interval.

Then, find the area of each rectangle by multiplying its height by its base (which happens to be 1 right now), and add them up to estimate the area. In this case, we get that the area is less than approximately 5.15.



interval	max of Sqrt(x)	Upper area	Do the same thing for the inner rectangles, and get the table at the right as a lower bound for the area.	interval	min of Sqrt(x)	Lower area
[1,2]	1.41	1.41		[1,2]	1.00	1.00
[2,3]	1.73	1.73		[2,3]	1.41	1.41
[3,4]	2.00	2.00		[3,4]	1.73	1.73
	total	5.15			total	4.15

We conclude that

$$4.15 \leq A \leq 5.15.$$

This is a better estimate. We can do even better by taking smaller and smaller rectangles.

Task: Do this again, with rectangles of length 0.2. Since the whole interval is of length 3, so you will be using 15 inner rectangles and 15 outer rectangles.

Hints: A spreadsheet is a good tool for this.

Since this function is monotone increasing, the minimum on each interval is always at the left-hand endpoint of the interval, and the maximum is on the right. This doesn't always happen.

Epiphany:

The Fundamental Theorem of Calculus tells us that, if $f(x)$ is the derivative of $F(x)$ on an interval $[a,b]$, then the area under $f(x)$ between a and b has is equal to $F(b) - F(a)$.

Note that if the graph of $f(x)$ below the $x -$ axis is counted as *negative* area.

Task: Find the appropriate $F(x)$ for the square root function. Find $F(4) - F(1)$. Check that this value is consistent with the estimate that you got above.

Integrals:

This value, the area under the curve $f(x)$ on an interval $[a, b]$ is called the *definite integral* of the function on the interval. It is denoted $\int_a^b f(x) dx$. Here, we have

estimated, then evaluated $\int_1^4 \sqrt{x} dx$

Figure out how to do this evaluation using Derive.

Task: Do this all again, that is estimate, evaluate, and evaluate using Derive, the value of $\int_0^2 (x-2)^2 dx$. For your estimate, use sub-intervals of length 0.2.

Things to hand in:

What you hand in must include.

1. Four tables
2. Progressively better estimates of $\int_1^4 \sqrt{x} dx$, with explanations of how you found them. You can't do this without pictures.
3. How you used the Fundamental Theorem to evaluate $\int_1^4 \sqrt{x} dx$
4. How you used Derive to evaluate $\int_1^4 \sqrt{x} dx$
5. Repeating steps 2, 3 and 4 with $f(x) = (x-2)^2$ on the interval $[0,2]$.

These parts, of course, must be embedded in a good, well-written technical report, with goals, summary, details and conclusions.