Numerical Analysis – Lab 11
The Hanging Chain

Goals

The goals of this lab are

1. to introduce a different kind of optimization problem
2. to do a problem that requires more calculations and, maybe, a little less commentary
3. to do a problem that will require you to invent some new techniques and ideas.

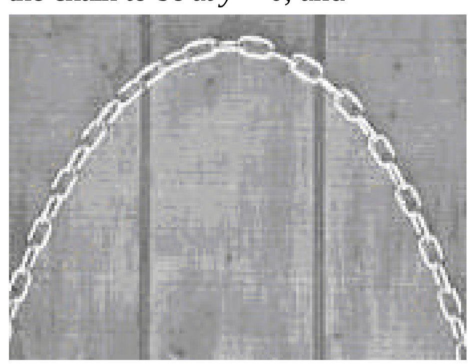
Preliminaries

Imagine a 15 foot chain (or cord or rope or telephone wire) fixed points ten feet apart. It will loop down between its ends. Our problem is to try to figure out the shape of that loop.

Our basic strategy will be to do a piecewise linear approximation.

Meanwhile, gravity will try to pull the center of gravity of the chain as low as possible. So, we will try to find a continuous, piecewise linear curve of length 15 that makes the center of gravity as low as possible.

To avoid negative numbers, let's take the ends of the chain to be at $y = 0$, and between the endpoints let y measure, as a positive number, the distance the chain is below the endpoints. So, although the chain will hang as in the upper picture, our graph will show it upside down, as in the next picture.



Symmetry

"Obviously" the curve should be symmetric. Be clever. Set up the bottom half of your spreadsheet so it takes the values of y from the top of the spreadsheet, so you don't have to keep re-copying or re-calculating those values.

Center of Gravity

The center of gravity of a curve has an x-coordinate and a y-coordinate. If the curve has an axis of symmetry (as this one does), then the center of gravity is on that axis.

Question 1: Explain why this means that the x – coordinate of the center of gravity must be 5.

If a segment is "uniform density" then the center of gravity of a line segment is the midpoint of that segment and the mass of the segment is the length of the segment times its density.

The center of gravity of several segments is the weighted average of the centers of each segment. That is, multiply the coordinates of the center of gravity of each segment by the length of that segment. Add up the products, and divide by the total length. Since we answered Question 1, we already know the x-coordinate for the center of gravity, so we only have to do this for the y-coordinates.

Question 2: Explain clearly how this "center of gravity" calculation works. You might want to refer to Archimedes' Law of the Lever in your explanation. Express the formula using Equation Editor and nice Sigma notation.

Restating the problem

We want to find the continuous curve $f(x)$

1. $f(0) = f(10) = 0$ endpoint conditions
2. length of f is 15 arc length condition
3. maximize the y – coordinate of the center of gravity of f.

From Question 1, we know that our solution will have the additional property that

4. f is symmetric about the axis $x = 5$.

Getting Started

Let's start with a two-segment model. It makes sense to take the segments to go from 0 to 5 and from 5 to 10, as shown in the *x-position* column of the spreadsheet below.

x-position	y-position	length	center	product	cum sum
0	0				0
5	5	7.071068	2.5	17.67767	17.67767
10	0	7.071068	2.5	17.67767	35.35534
	total length	14.14214			

The y positions at $x = 0$ and $x = 10$ have to be zero, by Property 1 of the problem. As a first guess, I took $f(5) = 5$. This is shown in the second column.

The third column gives the length of each segment, as calculated using the Pythagorean theorem. It looks something like =Sqrt((A3-A2)^2+(B3-B2)^2).

The fourth column gives the y coordinate of the center of gravity of the segment. It is just the average of the y-coordinates of the endpoints.

The column labeled *product* is the length times the y – coordinate of the center. The last column accumulates the sums of the products.

I didn't calculate the y-coordinate of the center of gravity. To do that, I would have divided the last accumulated sum, 35.35534, by the total length of the curve, which *should* be 15. This is the quotient we want to maximize.

Also, below the main part of the table, I sum the lengths of the individual segments to get the total length of the curve. This should be 15, but it is only 14.14214. This violates Property 2 of the problem.

Optimizing

So, I have to tinker with the values. In this simple case, the only value I'm allowed to change is $f(5)$. After a while, I find that $f(5) = 5.59$ makes arc length equal to 15, to within 0.01.

x-position	y-position	length	center	product	cum sum
0	0				0
5	5.59	7.499873	2.795	20.96215	20.96215
10	0	7.499873	2.795	20.96215	41.92429
	total length	14.99975		center	2.795

Notice I added the y coordinate of the center of gravity, labeled *center*, and found by dividing the last cumulative sum, 41.92429, by the total length of the curve, now 14.99975. This is the value we want to maximize.

I should graph this curve, but I didn't. It's just a triangle.

The next step

Two pieces aren't enough to give us a very good idea of the shape of the curve. Let's try four pieces, beginning with the spreadsheet below.

x-position	y-position	length	center	product	cum sum
0	0				0
2.5	2.795	3.749937	1.3975	5.240536	5.240536
5	5.59	3.749937	4.1925	15.72161	20.96215
7.5	2.795	3.749937	4.1925	15.72161	36.68376
10	0	3.749937	1.3975	5.240536	41.92429
	total length	14.99975		center	2.795

I have added two new points to the previous spreadsheet, at $x = 2.5$ and $x = 7.5$, and I have started them both with y values of 2.795, exactly half way between the values of 0 and 5.59 that I had from my two-piece solution. Note that the arc length and the center stay the same.

Now, it happens that if I decrease the value at $x = 5$, that decreases my center, which is bad, but it also decreases my total length and lets me lower my values at $x = 2.5$ and $x = 7.5$, which increases my total length and increases my center again. I quickly got:

x-position	y-position	length	center	product	cum sum
0	0				0
2.5	4.31	4.98258	2.155	10.73746	10.73746
5	4.62	2.519147	4.465	11.24799	21.98545
7.5	4.31	2.519147	4.465	11.24799	33.23344
10	0	4.98258	2.155	10.73746	43.9709
	total length	15.00345		center	2.930719

Note that I've increased my center of gravity (good) from 2.79 to 2.93. This is better. With some more tinkering, I managed to get a center of gravity above 2.95.

Continuing

Extend your spreadsheet to at least ten segments. Can you get your center of gravity above 3.00?

Is it a parabola?

The chain hangs in a shape that looks like a parabola. You can flip it over and put it on the x-axis by subtracting each value from the value at $x = 5$. Then you can shift it to the origin by changing the x values so they go from –5 to 5 instead of from 0 to 10. None of this changes the arc length.

Now you'd like to compare this to the parabola between –5 and 5 with arc length 15. The parabola has form $f(x) = ax^2$ which has derivative $f'(x) = 2x$. The arc length of a function between –5 and 5 is $L = \int_{-5}^{5} \sqrt{1+(f'(x))^2}\,dx$. So, to find the parabola with arc length 15 between –5 and 5, you have to solve for a, $\int_{-5}^{5} \sqrt{1+(2ax)^2}\,dx = 15$.

Question 3: Explain how you solved this equation for a.

Question 4: Compare the center of gravity of this parabola to the center of gravity you got for your curve. (If the parabola is better, then you didn't get the best answer above. Work on that some more.

Conclusion

Is the hanging chain curve a parabola? Why or why not?

Write it all up

Write it up carefully, following (more or less) the guidelines. Be clear, complete and concise. Give both technical conclusions, comparing the methods, and "managerial" conclusions, describing whether or not the goals were accomplished. The lab is due in two weeks.