Numerical Analysis – Lab 12
Differential equations I

Goals

The goals of this lab are:

1. to use Euler's method to solve first order differential equations given as slope fields,
2. to compare the numerical solutions with the exact analytical solutions found by other means,
3. to examine some of the limitations of Euler's method,
4. to speculate on how to overcome some of those limitations.

Preliminaries

We have learned two general formulations of first order differential equations:

Explicit form: $y' = \frac{dy}{dx} = f(x,y)$

Implicit form: $F(x,y,y') = 0$

In class, we showed how to solve such differential equations numerically, using slope fields. Now it is time to do some of the actual work. All of our examples today will be given in explicit form.

So that we can compare our numerical examples with the actual answers, we should use slope fields for which we can find explicit solutions. Let's introduce our three slope fields:

I. $y' = f(x,y) = \frac{2y}{x}$.

This can be solved by separation of variables:

$\frac{y'}{y} = \frac{2}{x}$. Integrate both sides and get

$\ln y = 2\ln x + c$. Take a such that $c = \ln a$, and use the rules of logarithms to get

$\ln y = \ln ax^2$ so

$y = ax^2$.

(There is a small rather technical calculus mistake in this. You can have two extra points if you can spot the one I'm thinking of.)

This is a family of parabolas each passing through the origin, and for a given point (x,y), the parabola through that point has $a = \frac{y}{x^2}$. Note that certain problems ensue if ever $x = 0$.

Note that if $a > 0$, then the corresponding parabolas curve upwards, and if $a < 0$, they curve downwards.

II. $y' = g(x,y) = 2x$

This, too can be solved by separation of variables.

Task 1: Use the techniques above to show that this, too, is a family of parabolas, but this time each has a different *y* intercept, *c*. For a given point (*x*,*y*), the parabola through that point has $c = y - x^2$.

III. $y' = h(x,y) = x - y$

This is a little trickier to solve, but its solution is $y = x + ce^{-x} - 1$, where *c* is a constant of integration and is determined by the initial values.

Task 2: Show that the solution given is in fact a solution to the differential equation, and give a value for *c* in terms of the initial value.

IV. I, meanwhile, will use yet another example, $y' = k(x,y) = -y$. I know (and you should check it) that the solution to this is $y = ce^{-x}$, where *c* is a constant of integration that can be determined from the initial values.

Euler's method

Euler's method is based on a first-order Taylor series approximation. If you know *x* and *y*, then your differential equation tells you *y'*. Using this value of *y'*, Taylor tells us that at a nearby point $x + h$, the value for *y* is approximately $y + hy'$.

I designed the spreadsheet below, using initial values $x = 0$, $y = 1$:

initial values	x =	0
	y =	1
step size	h =	0.1
	k(x,y) = -y	

n	x	y	y'	new y
0	0	1	-1	0.9
1	0.1	0.9	-0.9	0.81
2	0.2	0.81	-0.81	0.729
3	0.3	0.729	-0.729	0.6561
4	0.4	0.6561	-0.6561	0.59049
5	0.5	0.59049	-0.59049	0.531441
6	0.6	0.531441	-0.53144	0.478297
7	0.7	0.478297	-0.4783	0.430467
8	0.8	0.430467	-0.43047	0.38742
9	0.9	0.38742	-0.38742	0.348678
10	1	0.348678	-0.34868	

The first column, n, is just a column of step numbers.

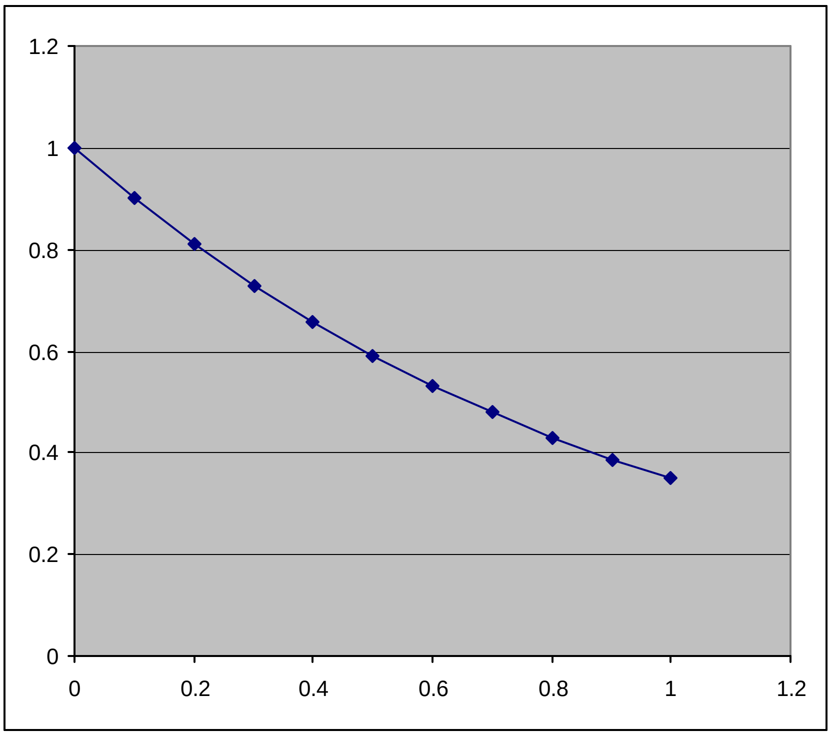
The second column gives my x values. Each value is h greater than the value above it.

The third column gives my y values. The first value is copied from my initial conditions. Each subsequent value is copied from the *new* y value from the row above.

The fourth column gives my y' values, using the explicit formula given for this problem: $y' = k(x,y) = -y$. Really, this is the only column you have to change much.

The fifth column is my *new* y value, and is calculated as $y + hy'$, referring to the values of y and y' given in each row and to the value of h at the top of the page.

I got the graph below:



Using $h = 0.1$, I estimate that $y(1) = 0.348678$. This is a little more than 5% smaller than the more correct value of 0.367879

Using a 100-line spreadsheet and a value of $h = 0.01$, I get a much more accurate estimate that $y(1) = 0.366032$. This is about half a percent low.

This suggests that, in general, smaller values of h give more accurate results.

Your jobs:

Task 3: Use your first slope field, f, with $h = 0.1$ and initial values $x = 1$, $y = 1$ and with initial values $x = 1$, $y = -1$, to find the value of the solutions at $x = 2$. Compare your estimates to the correct values ($y = 4$, in the first case and $y = -4$ in the second. Show why these are the correct values.)

Try to formulate a rule that tells when your estimate will be high and when it will be low.

Task 4: Use your second slope field, with initial values $x = 1$, $y = 0$, to study the effects that various values of h have on your estimate of the value of y when $x = 3$. Use a minimum of 3 values of h.

Task 5: Using your third slope field and starting with $x = 0$, look at the effects of various values of y, in the range of – 5 to 5, have on the shapes of your curves.

Task 6: Using your first slope field and $h = 0.1$, and the initial conditions $x = -1$, y = 1, you see from the Preliminaries that $a = 1$ and $y = x^2$, so $y(1) = 1$. But when you run your spreadsheet on this, you get an answer that is quite different. Can you explain this big error?

Task 7: If the curve through the point (x_0, y_0) passes through the point (x_1, y_1), then the curve through the point. Using your second slope field, with initial conditions (0,0) and $h = 0.1$, estimate y when $x = 1$. Use that value of y and $h = -0.1$ to estimate y when $x = 0$. If everything worked perfectly, you'd get $y = 0$. How much did you miss by?

Is this consistent with your results from Task 3?

Task 8: Can you think of a way to use your results from Task 7 to improve Euler's method and get better estimates for only three times as much work?

Write it all up.

Material for this lab is from Chapter 5 section 2 of your textbook. You will find it easier if you read and understand that material.

This lab is supposed to be relatively straightforward. You can use almost the same spreadsheet for all the Tasks. All you have to change are the parameters (initial values and step size) the formulas for y' in column 4, and sometimes the number of rows.

Do as much of this as you can in the lab. Talk about it there, and ask me about it.

If you are tempted to give me a spreadsheet with $n > 10$, cut out all but the first three and last three lines of data.

The lab is due in two weeks.