Numerical Analysis – Lab 13
Differential equations II

Goals

The goals of this lab are:

1. to improve on Euler's method to solve first order differential equations given as slope fields,
2. to have a short one

Preliminaries

Last time, we worked with $y' = f(x, y) = \frac{2y}{x}$. We (meaning I) showed that the general solution to this is given by $y = ax^2$, and that, given an initial value (x_0, y_0) we can find that taking $a = \frac{y_0}{x_0^2}$ gives the particular solution passing through that initial value.

We used Euler's method, and, for a given step size, *h*, we took $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$.

Higher order methods

With our extensive numerical analysis experience, we instantly recognize $y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$ as a first-order Taylor series approximation, and know that we could get more accuracy with only a little extra work by taking a second-, or even third- or fourth-order approximation.

Let's try a second-order. We have a formula for *y*'. Let's differentiate to find *y*".

$$y' = \frac{2y}{x}, \text{ so, by the quotient rule,}$$

$$y'' = \frac{2xy' - 2y}{x^2}$$

If we wanted to, we could substitute $y' = f(x, y) = \frac{2y}{x}$, but I don't feel like it.

Now, a second-order Taylor approximation gives: $y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n$. Rather than do all the substitutions, we can do the work in a spreadsheet, as follows:

initial x =	1				
y =	1				
h =	0.1				
				y"=2xy'-2y	
n	x	y	y'=2y/x	/x^2	new y
0	1	1	2	2	1.21
1	1.1	1.21	2.2	2	1.44
2	1.2	1.44	2.4	2	1.69
3	1.3	1.69	2.6	2	1.96
4	1.4	1.96	2.8	2	2.25
5	1.5	2.25	3	2	2.56
6	1.6	2.56	3.2	2	2.89
7	1.7	2.89	3.4	2	3.24
8	1.8	3.24	3.6	2	3.61
9	1.9	3.61	3.8	2	4
10	2	4			

Since the solution to this differential equation is a second degree polynomial, a second order Taylor approximation gives an exact answer. That's not quite fair, but it shows how the method works.

Your jobs:

Task 1: You did exactly this slope field in Lab 12. Compare your results here with your results there.

Task 2: Compare this method with your results on the other slope fields from Lab 12 as well.

Task 3: For the slope field that seems to do the worst, do a third-order method and see how much (if any) that improves your estimates.

Write it all up.

Material for this lab is from Chapter 5 section 2 of your textbook. You will find it easier if you read and understand that material.

Do as much of this as you can in the lab. Talk about it there, and ask me about it.

If you are tempted to give me a spreadsheet with $n > 10$, cut out all but the first three and last three lines of data.

The lab is due in two weeks.