

Numerical Analysis – Lab 15

Rational approximations

Goals

The goals of this lab are:

1. to review the properties of rational functions
2. to learn a technique to approximate functions
3. to develop means to measure how good an approximation is.

Preliminaries

A rational function is a quotient of two polynomials. In Precalculus, we were fond of asking you to sketch such functions by finding roots, vertical and horizontal asymptotes. If $r(x)$ is a rational function, it can be given by a quotient of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1x + p_2x^2 + \dots + p_nx^n}{q_0 + q_1x + q_2x^2 + \dots + q_mx^m}.$$

Note that there may be different ways to write r as a quotient, but the number of ways is greatly reduced if we assume that p and q are in lowest terms and that the last coefficients p_n, q_n are not zero. If the rational function does not have an asymptote at $x = 0$, the representation becomes unique if we take $q_0 = 1$.

In some applications (Abstract Algebra, for example) we would write the set of all polynomials with real numbers as coefficients as $\mathbb{R}[X]$. In a similar notation, we write the set of rational functions as $\mathbb{R}(X)$. The polynomials are seen to be a subset of the rational functions if we take $q(x) = 1$.

There are two different definitions of the word “degree” of a rational function. If you are doing abstract algebra, you use the definition $d(r) = n - m$, because that lets you have a nice formula that, (unless the zero function is involved) $d(r_1 \cdot r_2) = d(r_1) + d(r_2)$.

In numerical analysis, though, we use the other definition, $d(r) = m + n$. This makes the degree a measure of the complexity of your rational function, and the number of coefficients necessary to describe a rational function is $d(r) + 1$.

Measuring approximation

Suppose one function g is supposed to approximate another function f , on some closed interval $[a, b]$. We want to describe how well g approximates f . For our examples, we’ll take $f(x) = \ln(x)$, and $g(x) = (x-1) - \frac{1}{2}(x-1)^2$ on the interval $[1, 2]$. This function g , of course, is the degree 2 Taylor approximation of f at $x = 1$.

Way 1 – Maximum absolute error

$Max = \max_{x \in [a, b]} \{ |f(x) - g(x)| \}$ measures how big your biggest error might be.

Way 2 – Total absolute error

$TAE = \int_a^b |f(x) - g(x)| dx$ measures your total error, the area between the functions f and g on the interval $[a, b]$. It is easy to draw, and has some neat properties. If, say, h approximates g and g approximates f , there is a nice formula relating the three TAE's. Also, if you rearrange the letters, you can make "tea," and everyone knows how much Dr. Sandifer likes tea.

One can also form Mean absolute error by dividing TAE by the length of the interval.

Extra credit opportunity (2 points) – find the neat properties (they are inequalities) that I mentioned.

Way 3 – Total square error

$TSE = \int_a^b (f(x) - g(x))^2 dx$. Properties of this measurement are commonly studied in statistics. Mean square error is found by dividing this by the length of the interval. MSE is a lot more common than TSE, and, of all these measures, it is probably the most useful.

Task 1: Find Max, TAE, MAE, TSE and MSE for

$g(x) = (x-1) - \frac{1}{2}(x-1)^2$ as an approximator of $f(x) = \ln(x)$ on the interval $[1, 2]$. Find each to 4 sig figs.

Hint: Things are a little simpler if we replace x with $x - 1$. Then our function

$f(x) = \ln(x - 1)$ (a little more complicated) and $g(x) = x - \frac{x^2}{2}$ (a lot simpler), and the

Taylor series is $T(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \text{etc.}$ That's a whole lot simpler.

Task 2: Find Max, TAE, MAE, TSE and MSE again, using $g(x)$ the degree 5 Taylor approximation. Note that this has *six* coefficients, counting the zero constant term.

Rational approximations

The *Padé approximation technique* for rational approximation is described in Chapter 8.4, starting on page 517 of your textbook. Since polynomials cannot have

asymptotes, either horizontal or vertical, it is reasonable to hope that a rational approximation might be a better approximation than a polynomial one. Since $\ln(x)$ has a vertical asymptote at $x = 0$, this could matter here.

Suppose we want to approximate $f(x)$ with a rational function $r(x)$, with numerator $p(x)$ of degree n and denominator $q(x)$ degree m , with $q_0=1$. We get to pick $m + n + 1$ other coefficients in the rational function, so take the Taylor expansion of degree $m + n$, and call it $T(x)$. That has $m + n + 1$ coefficients, and it should be approximately true that

$$T(x) = \frac{p(x)}{q(x)}.$$

Make it so, and $p(x) \approx T(x) \cdot q(x)$

Write this as series, so that

$$(a_0 + a_1x + a_2x^2 + \dots + a_{m+n}x^{m+n})(1 + q_1x + q_2x^2 + \dots + q_mx^m) \approx (p_0 + p_1x + p_2x^2 + \dots + p_nx^n)$$

Multiply out and match coefficients. Note that you know all the a 's, and you're trying to find the p 's and q 's. We get

Constant term: $a_0 \cdot 1 = p_0$, so we can find p_0 immediately

Linear term: $a_0 \cdot q_1x + a_1x \cdot 1 = p_1x$.

Quadratic term: $a_0 \cdot q_2x^2 + a_1x \cdot q_1x + a_2x^2 \cdot 1 = p_2x^2$.

...

Last term $a_{m+n} = q_mx^m \cdot p_nx^n$

This gives a linear system of $m + n + 1$ equations with the same number of unknowns. When you solve the system, you get your rational approximation.

For example, for $f(x) = \ln(x - 1)$, we get a Taylor series expansion of

$T(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \text{etc.}$ If we take $n = 4$ and $m = 1$, we should use the degree 5 Taylor series approximation, so, in the notation above,

$$a_0 = 0, a_1 = -1, a_2 = \frac{-1}{2}, a_3 = \frac{1}{3}, a_4 = \frac{-1}{4}, a_5 = \frac{1}{5},$$

and the polynomial equation is

$$\left(0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right)(1 + q_1x) = (p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4)$$

Matching constant terms, we get

$$0 \cdot 1 = p_0$$

$$p_0 = 0$$

Matching linear terms, we get

$$0 \cdot q_1 x + x \cdot 1 = p_1 x$$

$$p_1 = 1$$

Matching quadratic terms, we get

$$x \cdot q_1 x - \frac{1}{2} x^2 = p_2 x^2$$

$$q_1 - \frac{1}{2} = p_2$$

Matching cubic terms, we get

$$\frac{-x^2}{2} \cdot q_1 x + \frac{x^3}{3} \cdot 1 = p_3 x^3$$

$$-\frac{1}{2} q_1 + \frac{1}{3} = p_3$$

Matching quartic terms, we get

$$\frac{-x^4}{4} + q_1 \frac{x^4}{3} = p_4 x^4$$

$$-\frac{1}{4} + \frac{1}{3} q_1 = p_4$$

Matching quintic terms we get

$$\frac{x^5}{5} - \frac{q_1 x^5}{4} = 0$$

$$\frac{1}{5} = \frac{q_1}{4}$$

$$q_1 = \frac{4}{5}$$

We stop here, with six equations, six unknowns, and solve the system. (It is very easy. Don't bother to try to do it by machine. Do it by hand.) We get

$$p_0 = 0, p_1 = 1, p_2 = \frac{3}{10}, p_3 = -\frac{1}{15}, p_4 = \frac{1}{60}, q_1 = \frac{4}{5},$$

and the rational approximation is

$$\ln(x) \approx \frac{0 + x + \frac{3}{10}x^2 - \frac{1}{15}x^3 + \frac{1}{60}x^4}{1 + \frac{4}{5}x}$$

We see from the table below of values of $\ln(x+1)$, for x between 0 and 1, that this approximation is rather poor, but it is better than the corresponding Taylor approximation.

x	$\ln(x+1)$	p(x)	q(x)	r(x)	T(x)
0	0.0000	0.0000	1.0000	0.0000	0.0000
0.1	0.0414	0.1096	1.0800	0.1015	0.0953
0.2	0.0792	0.2248	1.1600	0.1938	0.1827
0.3	0.1139	0.3452	1.2400	0.2784	0.2645
0.4	0.1461	0.4704	1.3200	0.3564	0.3434
0.5	0.1761	0.6000	1.4000	0.4286	0.4229
0.6	0.2041	0.7336	1.4800	0.4957	0.5076
0.7	0.2304	0.8708	1.5600	0.5582	0.6029
0.8	0.2553	1.0112	1.6400	0.6166	0.7162
0.9	0.2788	1.1544	1.7200	0.6712	0.8561
1	0.3010	1.3000	1.8000	0.7222	1.0333

Task 3: Check my work.

Task 4: Find the other Padé approximations of degree 5, (that is, $m + n = 5$. We've done (4,1) here. (5,0) is the Taylor approximation. You do (3,2), (2,3) and (1,4). You're welcome to try (0,5). It shouldn't be much extra work, but it shouldn't give a very good approximation, either. Maybe I'm wrong.)

Use the criteria above to pick which approximation is best.

Write it all up.

The lab is due in three weeks.