Numerical Analysis – Lab 18
Continued fractions

Goals

The goals of this lab are:

1. to examine the approximation of irrational numbers by rational fractions
2. to learn LaGrange's method for the approximation of roots

Preliminaries

Perhaps you have seen the mathematics problem. Solve for x:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \text{etc.}}}}}$$

It is one of Dr. Lubell's favorites. Most of you will be familiar with the trick, to see the "x in the x;" to substitute and get $x = 1 + \frac{1}{x}$, and solve for x. It turns out that x is the so-called "golden ratio."

Forms like this are called "continued fractions." Euler wrote about them in 1737, though they were discovered about 50 years earlier by an English nobleman named Brounker.

Task 1: Work it out.

Task 2: Do it again for the slightly different continued fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \text{etc.}}}}}$$

Continued fractions in which the "numerators" are all 1's are called "simple continued fractions." All the CF's we look at here will be simple.

This last continued fraction is often written as [1; 1, 2, 1, 2, 1, 2, …], where the whole number is written before the semicolon, and the denominators are separated by commas after the semicolon. The CF for the golden ratio would be written [1; 1, 1, 1, …] In general, we can write these things like $x = [a_0; a_1, a_2, a_3, a_4, \text{etc.}]$

Evaluating CF's

Suppose you know a continued fraction, like

$$y = \sqrt{7} = [2;1,1,1,4,1,1,1,4,1,1,1,4,...],$$

and you want to convert it to a fraction. There is an amazing trick, developed by Euler. Set up a table as follows:

		a_1	a_2	a_3	a_4	etc.
1	a_0	m				
0	1	n				

Our first approximation of √7 is just a_0, that is, 2.

Now, to find m, multiply $a_1 a_0$ and add 1. Use the same pattern to find n; multiply $a_1 \cdot 1$ and add 0. So, our y = √7, we start

		1	1	1	4	1
1	2	1*2+1 = 3	m			
0	1	1*1+0 = 1	n			

Our next approximation of √7 is 3/1 = 3.

Now, our next m follows the same pattern. Multiply a_2 by our old m and add the previous one. In this case, that is 1*3 + 2 = 5. Same pattern for n, n = 1*1 + 1 = 2. This makes our third approximation of √7 is 5/2 = 2.5.

The actual value is *always* between consecutive estimates, so we know that 2.5 < √7 < 3. Since, to five decimal places, √7 = 2.64575, we're not off track yet.

Continuing, we get

		1	1	1	4	1
1	2	3	5	8	37	etc.
0	1	1	2	3	14	etc.

This last estimate, 37/14 = 2.64706, accurate to about 0.002, or less than 0.1%. This is remarkable, because the denominator is only 14, and 2.646 as a decimal has a denominator of 1000.

Task 3: Use this technique to find the first ten rational approximations of the golden ratio, x = [1; 1, 1, 1, 1, 1, etc.]

Finding CF's

I notice in this morning's paper that someone named BRoberts of the Baltimore Orioles leads the American League in batting average, currently batting .434. Since we're between 10 and 15 games into the baseball season, and he probably gets 3 or 4 at bats per game, he's probably had between 30 and 60 at bats so far. Let's see if we can figure it out.

So, to three decimal places, x = 0.434 is a rational number, p/q, and probably 30 < q < 60. Note that 0.434 is exactly 217/500, but our batter can't have had 500 at bats already. This must be something rounded off. Our task is to find what.

Obviously, a_o = 0. How to find the next denominator? Follow the following algorithm:

Step 0: Take remainder = the fractional value of x, in this case, 0.434.
Step 1: If the remainder is zero, or very close to zero, then stop; you probably have a rational number.
Otherwise:
Step 2: Take quotient = 1/remainder.
Step 3: The next denominator is the integer part of quotient.
Step 4: The next remainder is the fractional part of the remainder.
Step 5: Go back to Step 1.

With our batting average:

Iteration 1:
remainder = 0.434
quotient = 1/0.434 = 2.304147
next denominator = 2
next remainder = 0.304147
(first approximation is x = ½ = .5)

Iteration 2:
remainder = 0.304147
quotient = 3.287879
next denominator = 3
next remainder = 0.287879
(second approximation is x = 3/7 = .429)

Iteration 3:
remainder = 0.287879
quotient = 3.473684
next denominator = 3
next remainder = 0 .473684
(third approximation is x = 10/23 = 0.4348, which would be reported as .435, not .434. Keep going.)

Iteration 4 gives next denominator = 2, next estimate x = 23/53, and x = 0.4339, which will be recorded as 0.434. This is probably correct.

Iteration 5 gives x = 217/500, which is exactly correct, but the denominator is way too big.

Conclusion: Roberts probably has 23 hits in 53 at bats.

Wait! There are two more columns in the newspaper, AB and H. They show that Roberts has 53 AB's and 23 H's! Exactly!

Task 4: Find the continued fraction expansions for all the square roots from 2 to 15. (skip 4 and 5). Find patterns. Make hypotheses. Test them on the square roots between 17 and 24.

Task 5: Find the CF expansion for *e*. Find patterns.

Task 6: Find the CF expansion for π. Is π = 22/7? Can you find 355/113? Find patterns.

LaGrange's method

We'll do an example. Take $f(x) = x^2 - 2x - 1 = 0$. We seek the smallest positive root. (How do you know, without graphing, that it has one? Work it out.)

Iteration 1: Substitute n = 0, 1, 2, etc. until we find the integer part of our root. We find that f(2) is negative, but f(3) is positive, so x is between 2 and 3.

Take a_0 to be that integer, in this case 2.

Now, we know that x is between 2 and 3, so x = 2 plus something between 0 and 1. Write it as $x = 2 + \frac{1}{y}$. Since 1/y is between 0 and 1, we know that $y > 1$. Substitute $x = 2 + \frac{1}{y}$ into f(x) and get $g(y) = f\left(2 + \frac{1}{y}\right) = \left(2 + \frac{1}{y}\right)^2 - 2\left(2 + \frac{1}{y}\right) - 1 = 0$

Simplify this last to get $-1 + \frac{2}{y} + \frac{1}{y^2} = 0$, so, multiplying by y^2, $-y^2 + 2y + 1 = 0$, or $y^2 - 2y - 1 = 0$. Now, if we can find a root of this, we know y, and we know that x = 2 + 1/ = 2 + 1/y, and we'll know x.

Iteration 2: Substitute n = 0, 1, 2, etc. until we find the integer part of our root. As before, we find y is between 2 and 3.

Take a_1 to be that integer, in this case 2. So, y is 2 + something small, and, just like before, take $y = 2 + 1/z$. As before, we know that $z > 1$. Make the substitution, $y = 2 + 1/z$, and get a new polynomial.

Etc. Find the root of that new polynomial (2 again), and know that z is between 2 and 3. Write $z = 2 + 1/w$, and repeat. Call the next one u, then that's far enough.

Look what's happened. $x = 2 + \frac{1}{y}$, but $y = 2 + \frac{1}{z}$, etc., so

$$x = 2 + \frac{1}{y} = 2 + \frac{1}{2 + \frac{1}{z}} = 2 + \frac{1}{2 + \frac{1}{2 + w}} \; 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{u}}} = \text{etc.}$$

We guess that x = [2; 2, 2, 2, 2, etc.]

Fact: The continued fractions of roots of quadratics always repeat, but like decimals, there may be some "junk" at the beginning before the cycle begins. This one repeated quickly.

Task 7: Try it yourself. Say we want the smallest positive root of $x^3 - 4x - 2$. I know that the smallest positive root is about 2.21. Use LaGrange's method to find a fraction that approximates the root to four decimal places. (Your denominator should turn out to be less than 50).

Write it up.

A task-by-task organization is probably appropriate. You'll need lots of Equation Editor. Due Wednesday, May 11, 2005.