Math 416 – Introduction to Abstract Algebra

Chapter 1 – Introduction to Groups

From last time:

A group is a set G with an operation $\circ$ such that

1) the operation $\circ$ is closed on G
   that is, if a and b are in G, then so is $a \circ b$
2) the operation $\circ$ is associative
   that is, $(a \circ b) \circ c = a \circ (b \circ c)$
3) there is an identity element in G. Call it e
   that is, $e \circ a = a \circ e = a$, for every a in G
4) every element a in G has an inverse, call it a^{-1}, such that $a \circ a^{-1} =\circ a^{-1} \circ a = e$

A group does not have to be commutative (we call it "abelian"). If it is, then it means

5) $a \circ b = b \circ a$

Examples need a set and an operation.

Examples:

Z, Q, R, C and +
Q*, Q+, R*, R+, C* and x

More examples:

2x2 matrices with determinant <>0, and matrix multiplication
(identity is the "identity matrix". That's why it has that name.)

2-dimensional vectors, with vector addition

2x2 matrices (all of them) and addition. (additive identity is matrix of all 0's)

Different examples:

Symmetries of a square:

4 rotations R-0, 90, 180 and 270, in degrees.

Operation: Ra ° Rb = R(a+b mod 360)

inverse of Ra is R(360-a)

4 flips, 2 across diagonals D and D', one across the horizontal H and one across the vertical axis

each flip is its own inverse.

8 elements total, R0, R90, R180, R270, H, V, D, D'

8x8 Operation Table or *Cayley table* (Sir Arthur Cayley, 1854, wrote over 900 mathematics articles.)

R0 is the identity

HR90 means first do H, then do R90.

HR90 = D

Whole table is filled in on page 33, but you can do calculations on the square itself.

Operation is *closed* because every entry in the table is one of the 8 operations we already know.

In Fact: Any symmetry can be build out of repeated R90s and Ds (or any other flips)

Trust me it's associative.

It's not commutative.

R90D = H, but DR90 = V

Anything with a symmetry has an underlying group.

Symmetry theory = theory of groups

Our group on the square is called the Dihedral Group on the square and named D_4.

Any regular n-gon has a dihedral group D_n.

no dihedral group is commutative

If you take away the flips, then your group is called a Cyclic group, and denoted C_n or Z_n.

every cyclic group is commutative

Cyclic groups are "subgroups" of their associated dihedral groups.