

Math 342

Homework # 5

Ch. 1.4 p 31 # 1, 2, 6, 7, 14

1. Verify that Fano's geometry and Young's geometry are both incidence geometries.

Part 1: Fano

The axioms of Fano geometry are

- Axiom 1 There exists at least one line
Axiom 2 There are exactly three points on every line
Axiom 3 Not all points are on the same line
Axiom 4 There is exactly one line on any two distinct points
 i.e. Through any two points there is a unique line
Axiom 5 There is at least one point on any two distinct lines
 i.e. Any two lines intersect at least at one point
 i.e. No two lines can be parallel

Using these (and the theorems we have *proved* about Fano geometry), in order to show it is an incidence geometry, we must prove:

- Property 1: For any two distinct points, there is a unique line that is on both of them.
Property 2: Any line has at least two different points on it.
Property 3: There are at least three different points.
Property 4: Not all points are on the same line.

- Property 1: For any two distinct points, there is a unique line that is on both of them.
Proof: This is Fano Axiom 4 (reworded). \checkmark

- Property 2: Any line has at least two different points on it.
Proof: Axiom 2 says "There are exactly three points on every line." Three is at least two. \checkmark

- Property 3: There are at least three different points.
Proof: Axiom 2 says "There are exactly three points on every line." and axiom 1 says "There exists at least one line." Together these mean there must be at least three different points. \checkmark

- Property 4: Not all points are on the same line.
Proof: This is exactly Fano axiom 3. \checkmark

Therefore, Fano geometry satisfies all four properties of Incidence geometry, so Fano geometry is an Incidence geometry.

QED

Part 2: Young

The axioms of Young's geometry are:

- Axiom 1 There exists at least one line
- Axiom 2 There are exactly three points on every line
- Axiom 3 Not all points are on the same line
- Axiom 4 There is exactly one line on any two distinct points
i.e. Through any two points there is a unique line
- Axiom 5 For each line l and each point P not on l , there exists exactly one line on P
and parallel to l .

This proof will resemble the proof for Fano.

Property 1: For any two distinct points, there is a unique line that is on both of them.

Proof: This is Young Axiom 4 (reworded). \checkmark

Property 2: Any line has at least two different points on it.

Proof: Axiom 2 says "There are exactly three points on every line." Three is at least two. \checkmark

Property 3: There are at least three different points.

Proof: Axiom 2 says "There are exactly three points on every line." and axiom 1 says "There exists at least one line." Together these mean there must be at least three different points. \checkmark

Property 4: Not all points are on the same line.

Proof: This is exactly Young axiom 3. \checkmark

Therefore, Young geometry satisfies all four properties of Incidence geometry, so Young geometry is an Incidence geometry.

2. Show that the axioms of incidence geometry are independent.

The axioms of incidence geometry are:

- Axiom 1: For any two distinct points, there is a unique line that is on both of them.
- Axiom 2: Any line has at least two different points on it.
- Axiom 3: There are at least three different points.
- Axiom 4: Not all points are on the same line.

The best way to show that they are independent is, for each axiom, to find a model that does not satisfy *that* axiom but *does* satisfy the other three.

There are thousands of examples. I choose some I like:

Part 1: Satisfies axioms 2, 3 and 4, but not axiom 1:

Three points, A, B and C
four lines, L1 = AB
 L2 = AB
 L3 = AC
 L4 = BC

Part 2: Satisfies axioms 1, 3 and 4, but not axiom 2:

Three points, A, B and C
four lines, L1 = A
 L2 = AB
 L3 = AC
 L4 = BC

Part 3: Satisfies axioms 1, 2 and 4, but not axiom 3:

One point, A
no lines at all

(I think there's only one other example for this: No points, no lines)

Part 4: Satisfies axioms 1, 2 and 3, but not axiom 4:

Three points, A, B and C
One line, ABC

6. An incidence geometry that exhibits the Euclidean parallel property is called an. Which of the geometries discussed in Sections 1. **affine geometry** 3 and 1.4 are affine geometries?

Four point geometry	yes
Fano geometry	no
Young geometry	yes
Euclidean geometry	yes

Others

four line geometry	no
three point geometry	no

7. Prove that in a finite affine geometry, all lines must contain the same number of points.

We get to assume three things:

- Incidence geometry (four axioms)
- Finite (i.e. finitely many points and lines)
- Affine (Euclidean parallel postulate)

Lemma: In an affine geometry, if three distinct lines l , m and n have l parallel to m and m parallel to n , then l is parallel to n .

Proof (by contradiction): Suppose l is parallel to m and m is parallel to n , but that l and n intersect at some point, call it P . Then l and n are two different lines through P and parallel to m , contradicting the Euclidean parallel postulate. \checkmark

We want to prove: Given any two lines, l and m , they must have the same number of points.

Proof: Let l and m be lines in this geometry.

Case 1: l and m are the same line. Then a line obviously has the same number of points at itself.

Case 2: l and m are different lines.

We plan to construct a 1-1 and onto function from the points on l to the points on m . Since l and m are not the same lines, there exists a point A that is on l but not on m , and another point B on m but not on l .

Let n be the (unique) line through A and B . (Incidence axiom 1)

Map A to B .

For any other point A' on l , find the (unique) line through A' that is parallel to n . Call that line h .

This new line h must intersect line m , else m and n would be two lines through B parallel to h , contradicting the lemma.

So, h intersects line m . Call the point of intersection B' , and map A' to B' .

The map is 1-1 because the line h can intersect l only one time, so only one point can map to B' .

The map is onto because, for any point B' on m , there is a line through B' , parallel to n and intersecting l . B' gets "hit" by the point where n intersects l .

QED

There may be an easier way to do this. Can you find it?

14. I didn't write up this solution.

- i) it is. ii) no. No line through C and J. iii) 9 iv) I skipped it
- v) 2 vi) 1, 2, 3, 4. vii) No. An affine geometry must be an incidence geometry and this violates Incidence axiom 1.