Infinitely many Pythagorean triples

Three numbers, *a, b* and *c,* that have the special property that $a^2 + b^2 = c^2$ are called a *Pythagorean triple* because of the Pythagorean theorem tells us that a triangle with sides of lengths *a, b* and *c* will be a right triangle. The numbers 3, 4 and 5 form a Pythagorean triple, as do 5, 12 and 13.

It certainly seems to be a special property, and there are a few obvious questions to ask.

First, how many different Pythagorean triples are there? Are there infinitely many?

Second, how can we find Pythagorean triples? Is there a way to find *all* of them?

For people who are in a hurry, there is a "killer" answer to both of these questions. We sometimes see that answer in a course on number theory, and we'll give that answer at the end of this essay. However, like a good mystery story, you miss all the "good stuff" if you go straight to the end, so we will take a slower, more scenic path to the answers.

Let's begin by making the question a little clearer. We already know that 3, 4 and 5 form a Pythagorean triple. Just a little thought shows that any "multiple" of this triple is also Pythagorean. Let's clarify this.

Let 3*n*, 4*n* and 5*n* be any multiple of the triple 3, 4 and 5 (where *n* is a positive integer. We're claiming that 3*n*, 4*n* and 5*n* is a Pythagorean triple.

$$\begin{aligned}(3n)^2 + (4\,n)^2 &= 9n^2 + 16n^2\\ &= (9+16)\,n^2\\ &= 25n^2\\ &= (5n)^2\end{aligned}$$

So, 3*n*, 4*n* and 5*n* is a Pythagorean triple, as we claimed. Moreover, since this is true for *any* positive integer *n*, and since each *n* gives us a different Pythagorean triple, we have shown that there are, indeed, infinitely many of them.

This is not satisfying, though. In a way, a 3-4-5 triangle isn't much different from its double, a 6-8-10 triangle, or its triple, a 9-12-15 triangle. They are all the same shape, just different scales. They are all similar to each other. They aren't different enough to be regarded as *different*.

So, let's call a triple *primitive* if its three elements are relatively prime. Our first triple, 3, 4 and 5, is primitive, and so is the triple 5, 12 and 13, but the multiples 6-8-10 and 9-12-15 are not primitive because they have common factors 2 and 3 respectively.

Exercise:

1. Show that in a primitive Pythagorean triple, one side is always even, the other is always odd, and the hypotenuse is always odd.

Knowing this, let's revise our questions and ask about *primitive* Pythagorean triples instead: Are there infinitely many *primitive* Pythagorean triples? How can we find them? Can we find all of them?

Let us turn our attention back to 1225 in the city of Pisa, now part of Italy. That's when a Leonardo of Pisa, perhaps better known by his nickname Fibonacci, published his *Liber quadratorum*, or *Book of Squares*. He wrote at least five books on mathematics between the years 1200 and 1230, and one of his other books is quite famous, the *Liber abaci*, or *Book of calculating*. That is where he introduces the Fibonacci numbers, the sequence that makes him best known today.

Right at the beginning of the *Liber quadratorum*, Leonardo writes:

> I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbes; for the unity is a square and from it is made the first square, namely 1; to this unity is added 3, making the second square, namely 4, with root 2; if to the sum is added the third odd number, namely 5, the third square is created, namely 9, with root 3; and thus sums of consecutive odd numbers and a sequence of squares always arise together in order.

Leonardo is describing a well-known property of sums of odd numbers:

$$
\begin{array}{lll}
1 & = 1 & = 1^2 \\
1 + 3 & = 4 & = 2^2 \\
1 + 3 + 5 & = 9 & = 3^2 \\
1 + 3 + 5 + 7 & = 16 & = 4^2 \\
1 + 3 + 5 + 7 + 9 & = 25 & = 5^2
\end{array}
$$

and, in general, the sum of the first n odd numbers,

$$1 + 3 + \ldots + (2n - 1) = n^2.$$

Leonardo doesn't give us a proof of this, but it is not a difficult proof if we use mathematical induction. We'll leave that to the reader, and instead give a "proof without words:"

1	2	3	4
1	2	3	5
1	2	4	6
1	3	5	7

$1 + 3 + 5 + 7 = 16 = 4^2$

Of course, the proof "without words" may take some words to explain how it works. Note the 1x1 cell in the lower left corner with a "1" in it. That represents the initial sum, $1 = 1^2$.

Wrapped around that, there is an L-shaped region of three cells containing the numbers 1, 2 and 3. That L, together with the original 1x1 square, form a 2x2 square, showing that $1 + 3 = 2x2 = 2^2$.

There is another L-shape wrapped around that, containing 5 more cells and forming a 3x3 square, showing that $1+ 3 + 5 = 3x3 = 3^2$.

We are supposed to "see" how each new L-shape adds the next odd number to the square formed by all the L-shapes that have come before it, and how it makes the next larger square. We will look at more "proofs without words" in another essay.

Leonardo immediately realizes that this gives him a way to find Pythagorean triples. (Leonardo knew about Pythagorean triples. Although Leonardo lived 800 years ago, Pythagoras lived about 1700 years before that.) He starts with any odd square number except 1. Let's do our example with 9.

$$9 = 3^2.$$

Then, look at the sum of all the odd numbers up to and including that odd square:

$$1 + 3 + 5 + 7 + 9.$$

By what Leonardo just taught us, this sum is a perfect square. Since there are five terms here, the square is $25 = 5^2$, so

$$1 + 3 + 5 + 7 + 9 = 5^2.$$

Let's add some parentheses:

$$(1 + 3 + 5 + 7) + 9 = 5^2.$$

The sum of the first four odd numbers is 4^2, and we picked 9 specially so that it would be a perfect square, so we get

$$4^2 + 3^2 = 5^2,$$

and we've found a (primitive) Pythagorean triple.