Series in the 60’s

Ed Sandifer
Western Connecticut State University
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**Official Abstract:**
Though about 10% of his entire production dealt with series, of 111 books and papers he wrote in the 1760’s only two were about series. We tell the short story of his paper on central trinomial coefficients, and the longer story of his paper about Bernoulli numbers.

**Subversive Abstract:**
Of over 800 books and articles by Euler, 81 or roughly 10% are about series. But of 111 books and articles he wrote in the 1760’s, only two are about series, and both of these are mostly unremarkable. Undeterred, we make remarks anyway.
Introduction

Euler wrote about 800 books and papers. An exact number is hard to define. The “official” number of entries in Eneström’s index is 866, but that includes a number of letters and unfinished manuscripts that Euler never expected to be published. Euler probably intended to finish some of the manuscripts, but others he had probably abandoned as dead-ends. Moreover, though most of Euler’s letters were simple communications, some were more like “open letters,” intended to be shared widely, so they were more like publications than private communication. Taking all of this into account, an estimate of “about 800” publications seems quite reasonable.

Euler wrote his first article in 1725, and it was published in 1726. He died in 1783, but papers intended for publication continued to appear until 1862, 79 years after his death. Below, we give a graph and a table describing the decades that Euler wrote 810 of his books and articles.

Of the 810 books and articles in this data set, the Editors of Euler’s Opera Omnia classified 81 of them, exactly 10%, as being about series, and so published them in volumes 14, 15 and 16 of Series I. The Editors used what some might think is an expanded definition of “series” that also includes infinite products and continued fractions. The timing of Euler’s work in series has a somewhat different shape than his work as a whole, as seen in the graphics below.
Euler’s interest in series seems to be declining through the heart of his career, in the 1740’s and 1750’s, to the point where he wrote only two papers on the subject in the whole of the 1760’s. One of those papers was on properties of the Bernoulli numbers, and the other, the one we discuss here, on properties of a particular series.

Central Trinomial Coefficients

E-326 Observationes analyticae
NCASP 11 (1765) 1767 p 124-143
OO I.15 p 50-69
written in 1763

The first of the two series papers from this decade is E-326, written in 1763 and titled Observationes analyticae, or “Analytical observations.” Euler plans to sum the middle terms of powers of quadratics, starting with the very simple quadratic, \(1 + x + xx\). He begins by listing the powers of \(1 + x + xx\):

\[
\begin{align*}
1 \\
1 + x + xx \\
1 + 2x + 3x^2 + 2x^3 + x^4 \\
1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6 \\
\text{etc.}
\end{align*}
\]

Now we look at the terms:
\(1, 1x, 3x^2, 7x^3, 19x^4, 51x^5, 141x^6, \text{etc.}\)
The *Encyclopedia of Integer Sequences*, [EIC] calls the coefficients “central trinomial coefficients.” Euler wants to know the rules that give these numbers.

He begins by rewriting
\[(1 + x + xx)^n = (x(1 + x) + 1)^n.\]
He expands the right hand side as a binomial, getting
\[x^n(1 + x)^n + \frac{n}{1} x^{n-1}(1 + x)^{n-1} + \frac{n(n-1)}{1\cdot2} x^{n-2}(1 + x)^{n-2} + \text{etc.}\]
This is just the Binomial Theorem. It would look more familiar if Euler had written this paper just a few years later, after he introduced an almost modern notation for binomial coefficients, writing \(\binom{n}{k}\) where we would usually write \(\binom{n}{k}\).

This last expression still contains binomials, so Euler steadfastly expands it again and combines like terms to find that the coefficient of \(x^n\) is
\[1 + \frac{n(n-1)}{1\cdot1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot1\cdot2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot2\cdot3\cdot1\cdot2\cdot3} + \text{etc.}\]
Armed with this formula, Euler calculates the first 12 terms of his sequence. If he had access to the online *Encyclopedia of Integer Sequences*, then in just a few moments he could have found over 20 terms:

1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 9539, 28601, 85483, 25653, 73789, 212941, 616227, 1787607, 5196627, 15134931, 44152809, 128996853, 377379369, 1105350729, 3241135527, 9513228123, 27948336381, 82176836301, 241813226151, …

Having found a direct formula for the central trinomial coefficients, and listing the first twelve coefficients, up to 73789, Euler begins a section mysteriously titled:

**EXEMPLUM MEMORABILE INDUCTIONIS FALLACIS**

Formulas are sometimes cumbersome, and this formula is particularly so. True to form, Euler sets out to find a recursive formula for these numbers. He writes his sequence in one row, the triple of the sequence, offset by one position, in the second row, and subtracts the first row from the second. It looks like this:

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>3</th>
<th>7</th>
<th>19</th>
<th>51</th>
<th>141</th>
<th>393</th>
<th>1107</th>
<th>3139</th>
<th>etc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>9</td>
<td>21</td>
<td>57</td>
<td>153</td>
<td>423</td>
<td>1179</td>
<td>3321</td>
<td>etc.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>30</td>
<td>72</td>
<td>182</td>
<td>etc.</td>
<td></td>
</tr>
</tbody>
</table>

Now Euler notices (*non sine ratione evenire videtur*, “not without thought it is seen to turn out that) all the numbers in this last row are the double of triangular numbers, and so have the form \(mm + m\), for various values of \(m\). Some people used to call these
products of consecutive integers of the form \(m(m + 1)\) oblong or Pronic numbers, but Euler does not use these terms.

So, what values of \(m\) give these particular values if \(mm + m\)? Euler calls these values of \(m\) the indices, and the indices go

\[1, 0, 1, 2, 3, 5, 8, 13, \text{etc.}\]

This is the Fibonacci sequence, starting just a little bit early, with first two terms 1 and 0, rather than the more familiar starting point 1 and 1. Actually, this apparently wasn’t called the Fibonacci sequence until the late 1800’s, but that wouldn’t keep Euler from knowing a lot about the sequence. In particular, he knows from his work on difference equations and generating functions that the \(n\)th term of this sequence is given by the formula

\[
\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n-2}
\]

Let us do something Euler couldn’t do, because subscripted sequences hadn’t been invented yet, and denote these values by \(f_n\).

From this, Euler can deduce a recursive formula. If we write the sums of the central trinomial terms as

\[1 + x + 3x^2 + 7x^3 + 19x^4 + \ldots + P\,x^n + Q\,x^{n+1} + \text{etc.}\]

then for the data in this table

\[3P - Q = (f_n + 1)f_n.\]

Euler also derives a direct formula for \(P\), and a second recursive formula that is homogeneous (i. e. one that does not involve \(n\).)

With these, we can find central trinomial coefficients quickly and easily, and we get

\[1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8955, 25675, 73945, \text{etc.}\]

But wait a minute! This isn’t the same sequence we started with! It is the same for the first nine terms, up to 3139, but for the tenth term, this has 8955 where there should be an 8953, and after that the differences become even larger. Now we see the meaning of the title of this section, which translates as “A notable example of false induction.” He had warned us. There really are two different sequences, each defined by reasonable and interesting patterns that agree for the first nine terms, and then become different.

Euler still has an article to finish, but nothing else this interesting happens. He finds the correct recursive relation directly from the formulas (so it is correct): if \(P\), \(Q\) and \(R\) are consecutive coefficients, then the \(n\)th term is given by the relation

\[R = Q + \frac{n+1}{n+2} (Q + 3P).\]
That done, he spends the rest of the article by investigating the central coefficients of powers of general quadratics of the form $a + bx + cxx$.

In 1753, Euler had written an E-256, *Specimen de usu observationum in mathesi pura*, “Example of the use of observation in pure mathematics.” It was an article about number theory, showing how experiments on integers of the form $a^2 + 2b^2$ led him to observe that such forms are closed under multiplication. This told him what to try to prove, and soon led to a proof of that and several related results.

Ten years later, he gives us a graphic illustration of the limits of observation, and that it shows mathematicians what might be true, not necessarily what is true.

Bernoulli numbers

E-393 De summis serierum numeros Bernoullianos involventium  
NCASP 14 (1769): I, 1770 p 129-167  
OO I.15 p 91-130  
written in 1768

The second of Euler’s series papers from the 1760’s is E-393, *De summis serierum numeros Bernoullianos involventium*, or “On sums of series involving Bernoulli numbers.” While the paper itself didn’t turn out to be all that interesting, the path that leads to the paper is fascinating. It is the story of the Bernoulli numbers.

Bernoulli numbers are a sequence of rational numbers that arise in a dazzling variety of applications in analysis, numerical analysis and number theory. When Charles Babbage designed the Analytical Engine in the 19th century, one of the most important tasks he hoped the Engine would perform was the calculation of Bernoulli numbers.

The first few Bernoulli numbers [K] are
After $B_1$ all Bernoulli numbers with odd index are zero, and the non-zero ones alternate in sign. They first appeared in 1713 in Jakob Bernoulli’s pioneering work on probability, *Ars Conjectandi*. Jakob Bernoulli (1654-1705) was the older brother of Johann Bernoulli (1667-1748), who was, in turn, Euler’s teacher and mentor at the University of Basel.

Sometimes people simply omit the Bernoulli numbers with odd index from the list, and write $B_k^*$ where we write $B_{2k}$. They, of course, must then make certain modifications to their formulas, and, in general, their formulas are a bit simpler.

Bernoulli was studying sums of powers of consecutive integers, like sums of squares,

$$1 + 4 + 9 + 16 + 25 = 55$$

or sums of cubes

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$$

In modern notation (Bernoulli did not use subscripts, nor did he use $\Sigma$ for summations or $!$ for factorials) Bernoulli found that

$$\sum_{k=1}^{n} k^p = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

If $n$ is large and $p$ is small, that means that the left hand side is a sum of a relative large number of relatively small powers, and if we know the necessary Bernoulli numbers then the sum on the right is simpler to evaluate than the sum on the left. Bernoulli himself is said [G+S] to have used this formula to find the sum of the tenth powers of numbers 1 to 1000 in less than eight minutes. The answer is a 32-digit number.

Bernoulli numbers arise in Taylor series in the expansion

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$
Bernoulli numbers are also involved in the expansions of several other functions, including $\tan x$, $\frac{x}{\sin x}$, $\log\left(\frac{\sin x}{x}\right)$ and others.

Euler encountered Bernoulli numbers in his great solution to the Basel problem when he showed that
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi^2}{6}, \]
though he did not recognize them at the time. In the same paper, Euler also evaluated
\[ \sum_{k=1}^{\infty} \frac{1}{k^n} \]
for the first several even values of $n$. Only later would he realize that these other sums involved the Bernoulli numbers. In fact, if $n$ is even, then
\[ \sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{2^n}{2(n!)} B_n \frac{\pi^n}{n!}. \]

Euler also failed to recognize the Bernoulli numbers in 1732 when he first did his work on the Euler-Maclaurin formula. Maclaurin also missed them when he discovered the formula independently in 1742. Again in modern form, the result says that for sufficiently smooth functions $f$, a series based on $f$ and an integral of $f$ are related by
\[ \sum_{k=1}^{n} f(k) = \int_{1}^{n} f(x) \, dx + \frac{f(1) + f(n)}{2} + \sum_{k=1}^{n} B_{2k} \frac{1}{2k!(2^{2k-1} - 2^{2k-1}(1))} + R_n(f, p), \]
where $R_n(f, p)$ is a remainder term that usually disappears rapidly as $p$ increases. [G+S]
The formula can be used either to estimate the series on the left knowing the integral on the right, or conversely, to estimate the integral by evaluating the series. Euler used the series on the left hand side of this formula in 1732 to estimate the values of infinite series and to find $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to six decimal places. This is also how he found the first properties of $\gamma$, the so-called Euler constant. Maclaurin used the other side of the formula to estimate the values of integrals from series.

The Bernoulli numbers are related to Euler’s constant $\gamma$ by
\[ \gamma = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \].
There is also an astonishing result due to Kummer [G+S] relating Bernoulli numbers to Fermat’s Last Theorem. Kummer noticed that a prime number $p$ is special if it does not divide the numerators of any of the Bernoulli numbers $B_2, B_4, B_6, \ldots B_{p-3}$. Such primes are now called regular. The first prime that is not regular is 37. Kummer showed that if $p$ is a regular prime, then Fermat’s Last Theorem, that $x^n + y^n = z^n$ has no non-trivial integer solutions, is true for $p$.

In 1921, Eric Temple Bell, author of the well-known popular mathematics history book *Men of Mathematics*, proved [B]:

*Theorem:* If $p$ is an odd prime which does not divide $4^{p-1} - 1$, then the numerator of $B_{2p-1}$ is divisible by $p$.

There must be thousands of such results, and Bernoulli numbers continue to be studied today. JSTOR reports over 150 “hits” on the key words “Bernoulli numbers” since 1990.

Let us turn now to 1755, when Euler published his *Institutiones calculi differentialis* [E212]. At that time, only a few of the results above were known, and their links to Bernoulli numbers were apparently not yet recognized. The results that were known seem to be:

1. Bernoulli’s own results on summing powers of integers. Bernoulli showed how this involved Bernoulli numbers, hence the name.
2. The Euler-Maclaurin summation formula.
3. Taylor series for various functions.
4. Euler’s evaluation of $\zeta(2n)$.

Then, through all the trees, Euler sees the forest. It must have been a wonderful feeling to see how so many different aspects of mathematics are linked through these mysterious Bernoulli numbers.

Euler devotes almost all of chapters 5 and 6 of Part 2 of his *Calculus differentialis* to results related to Bernoulli numbers, and on page 420 (page 321 of the *Opera Omnia* edition) he attributes them to Jakob Bernoulli and calls them *Bernoulli numbers*. Unfortunately, only Part 1 of the *Calculus differentialis* has been translated into English, so readers who want to enjoy it in Euler’s words must either brave the Latin or find a copy of the rare 1790 German translation.

Euler begins his chapter 5, “Investigation of the sums of series from their general term” with a quick treatment of Bernoulli’s results on summing sequences of powers. Then he repeats his own results from the 1730’s [E25] on the Euler-Maclaurin formula and gives the recursive relation on the coefficients in that formula. Euler doesn’t mention Maclaurin, so he is probably unaware of his work on the subject.
Then he shows how those coefficients arise from the Taylor series expansions of
\[ \frac{x}{1 - e^{-x}} \quad \text{and} \quad \frac{1}{2} \cot \left( \frac{1}{2} x \right) . \]

Eventually, after quite a bit of work, he lists the Bernoulli numbers, naming them after Bernoulli in the process, and shows how they are related to the coefficients in the Euler-Maclaurin formula.

This done, he extends occurrence of Bernoulli numbers in the expansion of
\[ \frac{1}{2} \cot \left( \frac{1}{2} x \right) \]
to the more general form \( \frac{\pi}{n} \cot \left( \frac{m\pi}{n} \right) \) and uses that to relate Bernoulli numbers to the values of \( \zeta(2n) \). To end the theoretical parts of his exposition, he gives some of the properties of the Bernoulli polynomials and notes that Bernoulli numbers grow faster than any geometric series.

Euler spends the rest of these two chapters doing applications of Bernoulli numbers, including calculating the Euler-Mascheroni constant, \( \gamma \), to 15 decimal places.

All this is rather unexpected in a textbook on differential calculus.

With this, Euler did not write again on Bernoulli numbers until 1768. In fact, in the intervening 13 years, he wrote only four papers on series. Besides the one on central trinomial coefficients that was the subject of last month’s column, he wrote one paper on approximating pi, one on trig functions, and one on continued fractions.

We have already said that the 1768 paper, E-393, “didn’t turn out to be all that interesting,” but it might be worth summarizing its results. Euler opens E-393 with a list of Bernoulli numbers and a list of the coefficients that arise in \( \zeta(2n) \), and shows how two lists are related. Then he gives his recursive relation on the zeta coefficients.

Then he leaps to the Euler-Maclaurin formula. Up to this point, most of the essay is just a new version of what he had presented in the *Calculus differentialis*. From here, though, he gives a different way to show the relation between the Bernoulli numbers and the expansion of \( \frac{1}{2} \cot \left( \frac{1}{2} x \right) \). Then he uses this same technique to give new relations between the Bernoulli numbers and a variety of other functions and numbers, including
\[ \frac{x e^y + e^{-y}}{2 e^y - e^{-y}} - \frac{1}{2} \quad \text{and} \quad \frac{1}{e^y - 1} . \]

Finally, he gives the values of integrals like \( \int_0^1 \frac{(\ln x)^n}{(1-x)} \, dx \), for \( n \) odd, in terms of the \( (n+1)^{th} \) Bernoulli number. None of this would have been appropriate to include in the *Calculus differentialis*.

Bernoulli numbers are still a bit mysterious. They appear frequently in Julian Havel’s recent book *Gamma*, about Euler’s constant, and people continue to discover new properties and to publish articles about them.
Simon Singh [S] quotes Andrew Wiles as describing the process of mathematical discovery with the colorful words “You enter the first room of the mansion and it’s completely dark. You stumble around bumping into the furniture but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it’s all illuminated.” It must have been something like this for Euler, when he saw how the “furniture” was arranged around the Bernoulli numbers.

In Euler’s time, though, light switches hadn’t yet been invented.

Conclusions

Sometimes Euler made mistakes. In E-326, he made a mistake in a fairly routine paper. He made it an interesting paper by sharing the mistake with us.

Almost all of Euler’s articles are connected to other articles. Sometimes, as in E-393, the connections are more interesting than the articles themselves.

One way or another, almost every Euler article is interesting. Our challenge is to figure out why.

References:


[G+S] Gourdon, Xavier and Pascal Sebah, Numbers, constants and computation, online at numbers.computation.free.fr/Constants/Constants.html, link to Constants, Miscellaneous, Bernoulli numbers, consulted July 25, 2005


