Leonhard Euler, Textbook Author

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ABSTRACT

Euler's four-volume 2500 page calculus text is often described as the origins or foundations of the modern calculus curriculum. This idea should not be accepted without some reflections. For example, few modern mathematics curricula include properties of elliptic integrals, as does Euler's *Calculus integralis*. His other textbooks are sometimes characterized incorrectly as well. We describe the content and intent of some of Euler's textbooks, and make some comparisons with the modern curriculum.
INTRODUCTION

Late last year, I was writing an encyclopedia entry on Euler. I wrote:

His four volume series of calculus books form the basis for the modern calculus curriculum, and was the first successful calculus textbook. It climaxes a complete series of mathematical textbooks, including arithmetic, algebra and the *Introductio in analysin infinitorum*, a textbook on the mathematics Euler thought was necessary to understand calculus.

Fortunately, by the time I had the chance to revise the article, I had read the *Calculus integralis*, and I was able to remove the words “form the basis for the modern calculus curriculum.”

This experience gives purpose to this article: to describe the contents and pedagogy of several of Euler’s textbooks, including his calculus texts, and to compare them to the modern curriculum.

Euler wrote textbooks on arithmetic, algebra, pre-calculus and differential and integral calculus. Vital statistics are given in the table below:

<table>
<thead>
<tr>
<th>Topic</th>
<th>Short title</th>
<th>Year(s) published</th>
<th>Eneström number(s)</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td><em>Einleitung zur Rechen-Kunst</em></td>
<td>1738, 1740</td>
<td>17, 35</td>
<td>277, 228</td>
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<tr>
<td>Algebra</td>
<td><em>Vollständige Anleitung zur Algebra</em></td>
<td>1770</td>
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<td>Pre-calculus</td>
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<tr>
<td>Differential calculus</td>
<td><em>Institutiones calculi differentialis</em></td>
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<td>212</td>
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</tr>
<tr>
<td>Integral calculus</td>
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<td>1768, 1769, 1770</td>
<td>342, 366, 385</td>
<td>542, 526, 639</td>
</tr>
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Another work, the three volumes of the *Letters to a German Princess*, should perhaps also be considered a textbook, but we omit it from this discussion for three reasons. First, it is not really a mathematics text. It is more of a popular science text. Second, there is nothing in the modern curriculum to which it compares. Third, and perhaps most important, we haven’t read it carefully, and would just as soon save it for another article.
Moreover, his great calculus of variations book, the *Methodus inveniendi* of 1744, reads much like a textbook, and perhaps Euler intended it as such. In any case, this is a big question and well beyond the scope of this article.

Excluding the *Letters to a German princess* and the *Methodus inveniendi* leaves us with plenty of material. The five textbooks fill ten volumes, and at 4172 pages of text, not counting prefices and tables, this curriculum is rather bulky, even by modern standards.

Euler wrote texts throughout his career. He wrote the mathematical texts at a pace of approximately one per decade, and, except for the *Algebra*, which appeared the same year as the *Calculus integralis*, he wrote them in their order in the curriculum, from arithmetic to integral calculus. We will discuss them in the order of the curriculum.

**RECHEN-KUNST**

*Einleitung zur Rechen-Kunst zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschaften in St. Petersbourg*, or the *Rechen-Kunst*, for short, was published anonymously, the first volume in 1738 and the second in 1740. Two volumes of a Russian edition appeared in 1740. The title page is shown in the illustration at the beginning of this article. When he delivered his *Éloge* in 1783, Euler’s son-in-law, Nicolas Fuss, revealed what by then must have been an open secret that Euler himself had written the books.

The short title, *Rechen-Kunst* is usually translated as “Introduction to Arithmetic.” The word *rechen* is the root of the English word “reckon.”

The first part, or *Erster Theil*, takes 277 pages in the German original, 162 pages in the *Opera Omnia*. The second is 228 pages in the original, 143 in the reprint. The first part is available on line at the site at the University of Bielefeld given in the references.

The first part contains nine chapters.

The first chapter, “Von der Arithmetic oder Rechenkunst überhaupt,” or “On arithmetic or *Rechenkunst* in general,” is about number names and place values. It is worth noting that, for Euler, it takes a million millions to make a billion, as in Germany today, not a thousand millions as in Canada and the US. The chapter consists of eleven numbered “paragraphs,” most of which contain a rule, an explanation of the rule and a few examples of how to use it. In some chapters, paragraphs also explain why the rule might be true, but Euler doesn’t do this in this first chapter.

The chapter ends with six examples, with answers, involving the age of the earth, King Solomon, Kaiser Augustus, the King of Assyria, Archimedes, and the number 1234567890987654321.

The remaining eight chapters of the first part explain the algorithms for addition, subtraction, multiplication and division, first of whole numbers, then of fractions. Most chapters have a structure similar to the first chapter. There are relatively few exercises at the end of each chapter, but what exercises there are are fairly difficult and quite literate.

The second part begins with 29 pages of tables of money and weights from around Europe and how the weights compare with each other. There follow five chapters on the arithmetic of such numbers with units. Euler calls them *bennanten Zahlen*, “named numbers.”
The first chapter is on “resolution and reduction.” Resolution is changing a larger unit into a smaller one, as pounds to ounces or rubles to kopeks. Reduction is the opposite process, changing a smaller one into a larger one, as pence into shillings and pounds.

The second chapter is about addition and subtraction of such named numbers, and, of course, depends heavily on the resolution and reduction of the previous chapter. The third chapter is about the multiplication and division of named numbers by whole numbers, the fourth on division of named numbers by other named numbers, and the last chapter is about multiplication and division of named numbers by fractions.

This last chapter has some unusual twists. For example, when multiplying by numbers like 5/24, Euler suggests that since \( \frac{5}{24} = \frac{1}{3} - \frac{1}{8} \), it is easier to divide first by three, then by eight, and subtract the results. The technique seems almost like an Egyptian unit fraction calculation. Euler is quite taken by the method and he does several examples.

This is a rather interesting and exciting conclusion to the *Rechen-Kunst*. Two things about the work are particularly striking. First, the book is written for a quite literate audience. The students learning from this book already know how to read well and have a good foundation in the classics. They know about King Solomon, Kaiser Augustus and Archimedes. They apparently don’t know even the most basic arithmetic, but they are expected to learn it by reading a lot of explanation and not very many examples or exercises. Rüdiger Thiele, in a private communication, [T] suggests that this may be because the book was written for what he calls a “Latin school,” the so called Gymnasium attached to the Academy of Sciences as part of its charter. Apparently students entering the Gymnasium already had a primary education that included reading, history and religious studies. Records of the St. Petersburg Academy [Ac] show that the students at the Gymnasium were more or less evenly split between children of the German community in St. Petersburg and children of the lesser Russian aristocracy, hence the need for editions of the *Rechen-Kunst* both in Russian and in German.

The second striking thing about the book is the small amount of material it covers, just the four basic arithmetic operations on positive integers, on fractions, and on “named numbers.” Let us compare the contents of the *Rechen-Kunst* with a popular early American arithmetic textbook by Daniel Adams. [Ad] We use an 1817 printing, though nearly identical versions were in print from 1801 until at least 1833. This edition is 216 pages long, and includes all of the material in the *Rechen-Kunst* except the multiplication and division of named numbers. On the other hand, Adams also includes decimal fractions, interest, both simple and compound, the Rule of Three, involution, evolution, square and cube roots, areas and volumes of circles and polygons, and a number of applications, like the rules of partnerships, examples for painters, glaziers and joiners, and more. He also does a bit of multiplication of named numbers when he shows how to calculate areas by multiplying lengths given in feet and inches. Moreover, Adams has far more exercises than Euler. A modern student would be much more comfortable learning from Adams than from Euler. Thiele [T] suggests that Euler modeled his *Rechen-Kunst* on Rudolff or on Stifel’s *Coss*. Adams seems to resemble arithmetics by Hodder or Cocker, published in England at about the same time as the *Rechen-Kunst*.  

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ALGEBRA

Having been surprised by the style and content of the *Rechen-Kunst*, should we expect the unexpected of his *Algebra* as well? We confess that we have not looked at the *Algebra* as closely as we have looked at some of the other textbooks, so the remarks that follow may be a bit shallow at times.

Euler’s *Algebra* was extremely popular. When it appeared in 1770, Euler was the pre-eminent mathematician and scientist in all of Europe, with a reputation akin to that of Albert Einstein, still the standard of genius 50 years after his death. The *Volständige Anleitung zur Algebra* was published in Russian in 1768 (two years before the German edition,) in Dutch in 1773, in French in 1774, in Latin in 1790, and in English in 1797. It was still the standard textbook in Germany a hundred years after it was first published, and the author has in his collection a copy that was apparently used as a textbook in the German speaking communities of upstate New York in the 1880’s. The original German edition was published in two volumes of 356 and 532 pages respectively, and given Eneström index numbers 387 and 388. The modern Springer edition is 462 pages, plus an appendix of 131 pages by LaGrange. We work from this edition.

The book is organized into two parts, what were the two volumes of the original German edition. Each part has paragraph numbers, 800 paragraphs in Part 1, 248 in Part 2. The German edition has three sections in Part I and two sections in Part II. In the English edition, the first section of Part II has been moved to Part I. Sections, in turn, are divided into at least a dozen chapters.

The *Algebra* doesn’t exactly pick up from where the *Rechen-Kunst* left off. The student is expected to know how the arithmetic of positive numbers and of fractions, but the material in the second part of the *Rechen-Kunst* is deemed irrelevant. In Volume 1 paragraph 6, Euler writes:

6. In Algebra, then, we consider only numbers which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of mathematics.

Four paragraphs later he writes:
10. All this is evident; and we have only to mention that in Algebra, in order to generalize numbers, we represent them by letters, as $a, b, c, d, \&c.$ …

From there, Euler moves rather quickly. On page 38 (of the Springer edition) we learn about square roots and that many of them are irrational. On page 42 we learn about “Impossible, or Imaginary Quantities.” Fractional exponents come on page 56, and logarithms on page 69. There are a good number of exercises. For example, on page 59, among “Questions for practice respecting surds.” we find:

19. Divide $a^2 - ad - b + d \sqrt{b}$ by $a - \sqrt{b}$

Ans. $a + \sqrt{b} - d$

Section II of Part I begins on page 76 and is titled “Of the different Methods of calculating Compound Quantities.” We would call them multinomials. The section seems innocent enough at the outset, but after we learn the binomial theorem, the next-to-last section applies it with fractional exponents to expand roots of binomials into infinite series, and the last uses negative exponents. We also learn the long division algorithm for polynomials, and, as a consequence, the formula for the sum of an infinite geometric series.

Much of the material that seemed to be missing in the Rechen-Kunst appears in Section III, decimals, arithmetic progressions, proportion and geometric progressions. Like many others of his day, Euler enjoys figurate numbers. One of his exercises, on page 145 is:

439. Question. A person bought a house, and he is asked how much he paid for it. He answers that the 365$^{th}$–gonal number of 12 is the number of crowns which it cost him.

In order to find this number, we make $m=365$, and $n=12$; and substituting these values in the general formula [given on the previous page], we find for the price of the house 23970 crowns.

Section IV of Part I of the English edition is Section I of Part II of the German edition, and the paragraph numbers were changed. It is a difficult chapter, titled “Of Algebraic Equations, and of the Resolution of those Equations.” Here, we learn how to solve simultaneous linear equations, with examples involving up to three variables, and polynomial equations up to the fourth degree. We also learn a variation on Newton’s method for approximating roots of equations.

There are a great many exercises, most of which are rather dry and computational, but one stands out as unusually witty:

612. Question 3. A mule and an ass were carrying burdens amounting to several hundred weight. The ass complained of his, and said to the mule, I need only one hundred weight of your load, to make
mine twice as heavy as yours; to which the mule answered, But if you give me a hundred weight of yours, I shall be loaded three times as much as you will be. How many hundred weight did each carry?

It is also interesting to note the variety of ways the familiar quadratic formula can be written. Euler gives the roots of \( x^2 + px = q \) with the formula \( x = -\frac{1}{2} p \pm \sqrt{\frac{1}{4} p^2 + q} \).

The last section of the text is devoted to what Euler calls “the analysis of indeterminate quantities,” and LaGrange calls “Diophantine Algebra.” Euler shows how to find integer or rational solutions to certain first, second and third degree equations. On page 405, he shows that Fermat’s Last Theorem is true in the case \( n = 4 \), a proof that Fermat himself had found. On page 449 he proves it in the case \( n = 3 \), the first known proof in that case. In the rest of the section, Euler reviews a great many of his own results in quadratic forms, showing what numbers can be represented as \( ax^2 + by^2 \) for given values of \( a \) and \( b \).

Euler’s Algebra is dramatically different from his Rechen-Kunst. It covers many times as much material in only about 50% more pages. It has many more exercises. A modern student might struggle with the volume of material, but that student would be fairly comfortable with the pedagogy. Where the Rechen-Kunst seems quite different from other texts, the Algebra seems to be part of a line of texts that goes through, say Bordon’s text in French, Davies’ English translation and adaptation, and then the late 19th and early 20th century texts that lead to those that are familiar today.

THE INTRODUCTIO

Euler’s Introductio in analysin infinitorum is regarded by many, Frederick Rickey, [R] for example, as Euler’s greatest work, indeed, one of the greatest single works in the history of mathematics.

Rickey enjoys comparing the table of contents of the Introductio with that of L’Hôpital’s differential calculus book published 50 years earlier. L’Hôpital’s full title is Analyse des infiniment petits, pour l’intelligence des lignes courbes, and, as the title suggests, L’Hôpital believes that calculus was useful to understand curves. A few of his chapter titles are:

II. The use of the calculus of differences to find the Tangents of all sorts of curved lines
III. The use of the calculus of differences to find the largest and smallest ordinates, to which are reduced the questions De maximis & minimis
VII. The use of the calculus of differences to find Caustics of refraction
VIII. The use of the calculus of differences to find the points of curved lines which touch an infinity of lines given in position, straight or curved.

To L’Hôpital, clearly the primary object is the curve. The equations that describe the curve are mere properties of the curve.
A few of the chapter titles from the first book of the *Introductio*, on the other
hand, are

I. On Functions in General
II. On the Transformation of Functions
IV. On the Development of Functions in Infinite Series
VI. On Exponentials and Logarithms
XII. On the Development of Real Rational Functions
XIII. On Recurrent Series
XVIII. On Continued Fractions

All of these chapters are about functions, not about curves. Even chapters like VI, XIII and XVIII that do not mention functions in their title deal with functions as the primary objects. Curves play no role.

The second volume of the *Introductio*, though, has an entirely different emphasis. It is about curves, and how to use analysis to understand curves. A few of its chapter titles are:

I. On Curves in General
III. On the Classification of Algebraic Curves by Orders
VIII. Concerning Asymptotes
XIV. On the Curvature of a Curve
XIX. On the Intersection of Curves
XXI. Transcendental Curves

It is widely accepted that during the 18th century, Mathematics underwent a transformation from a geometric approach, as characterized by L'Hôpital, to an analytic one, as in Book I of the *Introductio*. By 1755 when Euler published the *Calculus differentialis*, as we will see, he was almost completely committed to analysis over geometry, but in 1748, it seems he still had divided loyalties.

The mathematics of the *Introductio* itself is spectacular. It is the first important book to treat trigonometric functions as functions rather than as measurements. We see exponentials and logarithms for the first time in almost the same way we do them now, though Euler does use infinite numbers and infinitesimals in his calculations and derivations. Because Euler used the symbols $e$ and $\pi$ in the *Introductio* to denote those
constants, their use has become standard today. A merely superficial discussion of just
the highlights of the *Introductio* would be a long paper in itself. The reader, surely being
an interested reader, will want to read the *Introductio* itself. Fortunately, John Blanton’s
English translation is widely available [E101], and Fred Rickey has given us a guide to
reading the *Introductio*. [R]

The pedagogy of the *Introductio* is very different from the *Rechen-Kunst* or the
*Algebra*. There are no exercises, and there are relatively few examples. There are only
four examples, all in the chapter on exponentials and logarithms, that can fairly be
described as “word problems.”

It is hard to know how the student is expected to use the *Introductio* to learn. In
fact, this is related to one of the most difficult questions we can ask. The figures of the
past have left us a great many books and other texts. We read them today, but we can
only speculate how the people who wrote them and their contemporaries who studied
them actually read the texts. Still, a speculation is more interesting than a shrug, so we
will speculate.

The *Introductio* obviously requires a good deal more reader involvement than
today’s undergraduate textbooks require. The student is expected to follow the examples,
probably to invent variations on the problems and to work them out as well. This is
behavior more typically expected of graduate students today. It is how Serge Lang
expects his readers to behave in his famous exercise at the end of his chapter on
homology [L, p. 105]:

**EXERCISES**

Take any book on homological algebra, and prove all the
theorems without looking at the proofs given in that book.

**DIFFERENTIAL CALCULUS**

We now turn to the first of Euler’s calculus textbooks, the *Calculus differentialis*
of 1755. Euler’s calculus sequence, at 2587 pages, dwarfs the most prolix of today’s
texts. The four calculus volumes of the *Opera Omnia*, Series 1 volumes 10 to 13, occupy
6 ½ inches (16 cm) of my bookshelf.

As usual, the *Calculus differentialis* is divided into two parts, the first of 9
chapters, the second of 18, but this time they were published as a single, huge 880-page
volume rather than being split into two volumes. John Blanton [E212] has translated the
first part into English, and for this part we use his translation when discussing this part of
the work. We resort to the *Opera Omnia* for our comments on the second part.

To Euler, differential calculus is the calculus of differentials, not of derivatives.
In an example (p. 105) on the differentiation of transcendental functions, we find the
differential, not the derivative. It reads:
IV. If \( y = (\ln p)(\ln q) \) with \( p \) and \( q \) being any functions of \( x \), by the product rule given before

\[
dy = \frac{dp}{p} \ln q + \frac{dq}{q} \ln p.
\]

This, and eight other such examples, take the place of exercises in this and other chapters. Another delightful example of the chain rule appears on page 110:

If \( y = e^{ex} \), then

\[
dy = e^{ex} e^x e^x dx.
\]

Euler bases his entire approach on the relations between an arithmetic sequence of values of the independent variable, \( x, x + \omega, x + 2\omega, x + 3\omega, x + 4\omega, \text{etc.} \), and the corresponding values of the dependent variable, which, for want of subscript notation, he writes \( y, y', y'', y''', y''''\), etc. He defines first differences as \( \Delta y = y' - y, \Delta y' = y'' - y' \), etc., then second differences, and so forth. Only after he as developed as much theory of differences as he can does he take \( \omega \) to be an infinitely small number and get his differentials.

Some people criticize Euler for failing to pay attention to the rigorous foundations of the calculus. This criticism is unjustified. Euler takes great pains in his Preface to try to secure the foundations, with explanations like this (page viii):

To many who have discussed the rules of differential calculus, it has seemed that there is a distinction between absolutely nothing and a special order of quantities infinitely small, which do not quite vanish completely but retain a certain quantity that is indeed less than any assignable quantity. Concerning these, it is correctly objected that geometric rigor has been neglected. …

Euler goes on to try to justify the foundations of calculus. He did not neglect them; he merely failed in his attempts to make them rigorous.

In the Calculus differentialis, Euler’s conversion from Geometer to Analyst is complete. None of the chapter titles mention curves. There are no illustrations in the entire text, no differential triangles, no tangent lines. The objects of study are functions, given by rules or as series.
The pedagogy resembles that of the *Introductio*, with many examples, but no exercises. Many of the examples have a very familiar flavor, for example these from Part 2, the chapter on maxima and minima (p. 470)

**EXAMPLE 1**
To find the number which has the minimum ratio to its logarithm.

**EXAMPLE 2**
To find the number $x$ that makes the power $x^{1/x}$ a maximum.

In modern notation, we are asked first to minimize $\frac{x}{\ln x}$, then to maximize $\frac{1}{x^x}$. Note the archaic use of the notation $1:x$ for the ratio of 1 to $x$. Both problems are worked out in detail and answers given to at ten and fifteen decimal places respectively.

**INTEGRAL CALCULUS**

The three volumes of the *Calculus integralis* seem to be a classic work, more talked about than read. The volumes of the *Opera Omnia* in the Yale University Libraries seemed unread, and had not circulated since the 1970’s. Meanwhile, the *Introductio* volumes were well worn, and the *Algebra* had been used so often that it had been re-bound. Copies on the used book market are rare, and the copies that do appear are usually in very good condition, symptoms of a book bought as a trophy but not read. All three volumes did go through three Latin editions, but were translated only into German, and not into French, Dutch or English. Since print runs were typically very small, it is likely that there were as few as one or two thousand copies printed.

The three volumes have slightly different titles, describing their different contents.

E342  *Institutionum calculi integralis volumen primum in quo methodus integrandi a primis principiis usque ad integrationem aequationum differentialium primi gradus pertractatur*, 1768

E366  *Institutionum calculi integralis volumen secundum in quo methodus inveniendi functiones unius variabilis ex data relatione differentialium secundi altiorisve gradus pertractatur*, 1769

E385  *Institutionum calculi integralis volumen tertium in quo methodus inveniendi functiones duarum et plurium variabilium, ex data relatione differentialium cujusvis gradus pertractatur. Una cum appendice de calculo variationum et supplemento, evolutionem casuum prorsus singularium circa integrationem aequationum differentialium continente*. 1770.

These volumes are even more analytical than the *Calculus differentialis*, if that is possible. Euler begins:

Definition 1: Integral calculus is a method of finding from a given differential relation the quantity itself; and the operation by which this is done is called integration.
Integral calculus seems to be about differential equations and antiderivatives. It is not about curves or areas. Later he writes (page 9, paragraph 16)

Certain of what is treated in later parts of these elements has greatest use in Mechanics and in the doctrine of fluids. On account of this, only the very first rudiments of these things will be treated, and our second book of integral calculus will be sterile of such commentary, which, though it has long been necessary, very little has yet been done. When it is done, it will be seen to confer a great many scientific advances.

Euler draws his line even more sharply. His treatment of integral calculus will include no applications, either. This leaves just pure, sterile analysis.

The text itself is stark and fast moving. After a “praenotanda” on the nature of integral calculus, the curriculum is structured about a long sequence of “problems,” each followed by a solution, and usually several corollaries, scholions and examples. The first two volumes have 173 such problems, the numbers continuing from one volume to the next. The third volume has 88 problems, with twenty more problems (and seven illustrations!) in an appendix on the calculus of variations. A typical problem is number 79 from the first volume:

612. If \( \Pi : z \) denotes a function of \( z \) such that
\[
\Pi : z = \int \frac{dz}{\sqrt{A + Cz + Ez^4}},
\]
chosen so that it vanishes when \( z = 0 \), then to investigate the comparisons among such kinds of functions.

The kinds of “comparisons” Euler means are relations like
\[
\Pi : p + \Pi : q + \Pi : r = 0.
\]
Some readers will recognize this as one of the arc length sum formulas for elliptic integrals. As we mentioned, the text is fast moving.

This problem is followed by its solution three corollaries and two scholions. Work a few pages earlier makes it clear how such formulas are related to angle-sum identities.
for the trigonometric functions. Subsequent problems pose the same question for fourth degree polynomials that are not missing their terms of odd degree.

The textbook is clearly carefully planned so that the problems integrate so beautifully, but it also places a tremendous burden on the student, both to master the material and to relate it to its applications.

The Editors of the *Opera Omnia* appended to the second volume some notes by Mascheroni, titled *Adnotationes ad calculum integrale Euleri, In quibus nonnulla Problemata ab Eulero proposita resolvuntur*, that is “Annotations to Euler’s Integral calculus, in which several problems posed by Euler are resolved.” They are about 130 pages of notes, published in Pavia in two parts in 1790 and 1792. Mascheroni particularly focused on questions relating to $\gamma$, what is now called the Euler-Mascheroni constant.

The content of Mascheroni’s notes are surely interesting enough, and they made him worthy of having his name attached to the constant. It is even more interesting how they give us a glimpse of how Mascheroni read Euler, and, in turn, how Euler meant people to read his upper level texts. Mascheroni read Euler in much the same way Lang [L] urges his students to read homology: read the problem and try to solve it without looking at the solution given in the book. Then try to anticipate what the next problem will be and try to solve it before even reading the problem.

**CONCLUSIONS**

Though we did not state them, we began this paper with three implicit hypotheses:

1. Euler’s textbooks, particularly his calculus books, form the basis for the modern mathematical curriculum.
2. Euler’s textbooks were immensely influential.
3. Euler’s textbooks were the first that taught mathematics in the way it is taught today.

At the end of this paper, we find we must reject most of these hypotheses.

1. **Curriculum.**
   Only the *Algebra* and certain parts of the first book of the *Introductio* seem to be much like what we teach today in algebra and precalculus. The *Algebra* does seem to resemble more closely the algebra that was taught from, say, Bordon or Davies in the 19$^\text{th}$ century, but the curriculum has changed a good deal since then.

2. **Influence.**
   The *Introductio* and the *Algebra* were, indeed, very influential. Both were translated into many languages, and the *Algebra* was an important text for over a hundred years. The *Rechen-Kunst*, though, was obsolete just about as soon as it was written, and the two *Calculus* books were too difficult for most readers. They were not nearly so influential.
3. Pedagogy.

Euler’s textbooks were written to be studied and read, not to be explained or lectured about. Though this is the way those of us who have become professionals in the field learn most of our mathematics, it is seldom the way we teach. When we do teach like this, it is usually to graduate students, and it is often their first experience in such a learning mode. Euler even writes arithmetic to be learned like this.

There are a number of reasons we do not teach like this today. First, we would like to believe that teaching methods may have improved in 250 years. Second, students have so much more to learn today, and they have many more choices of what to try to learn. Third, we try to educate a very large segment of our populations in classrooms much larger than most in the 18th century. In 1750, Berlin had a population of about 1 million (one of the four word problems we mentioned from the Introductio). It would be astonishing if a hundred of them, that is one hundredth of a percent of them, knew calculus. With such an elite population of scholars, entry standards could be very high, much higher than we would accept today.

ACKNOWLEDGEMENTS

What I know of these things, I know because I have read the books themselves. There is no substitute for original sources. I would like to thank my university for giving me the liberty, and Yale University for giving me the access to the books and journals, many of them rare and priceless, that make such studies possible.

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